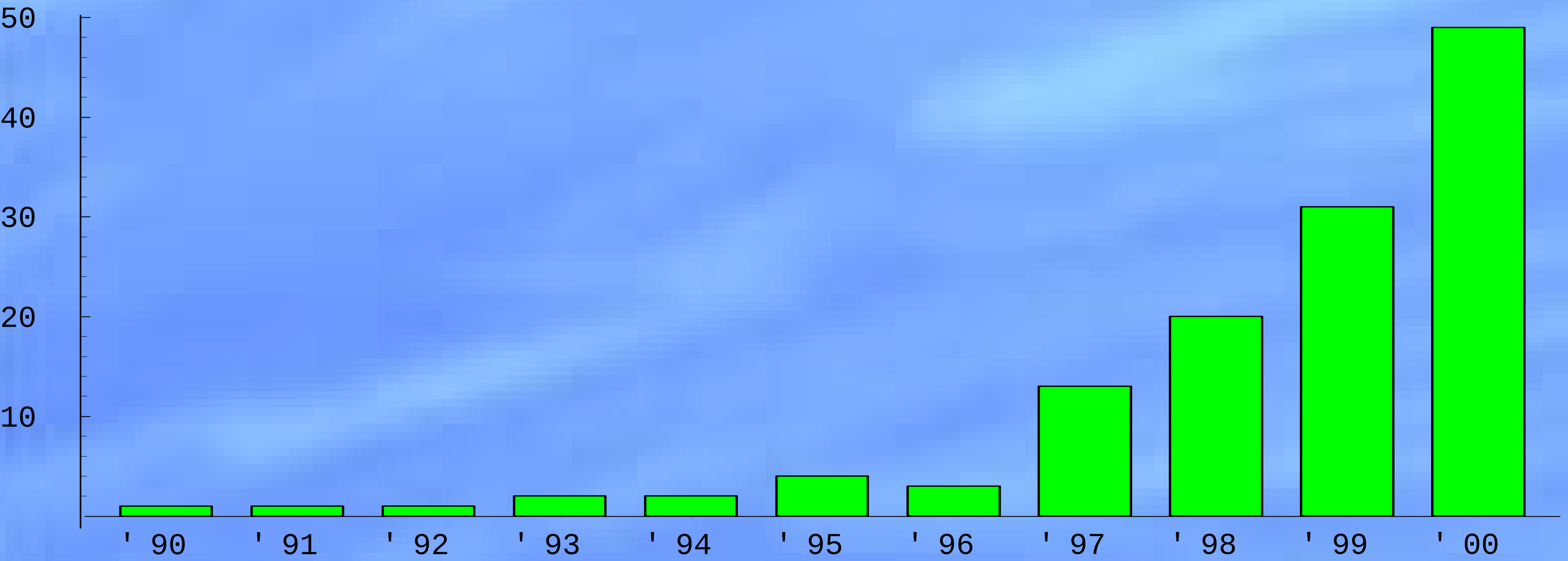
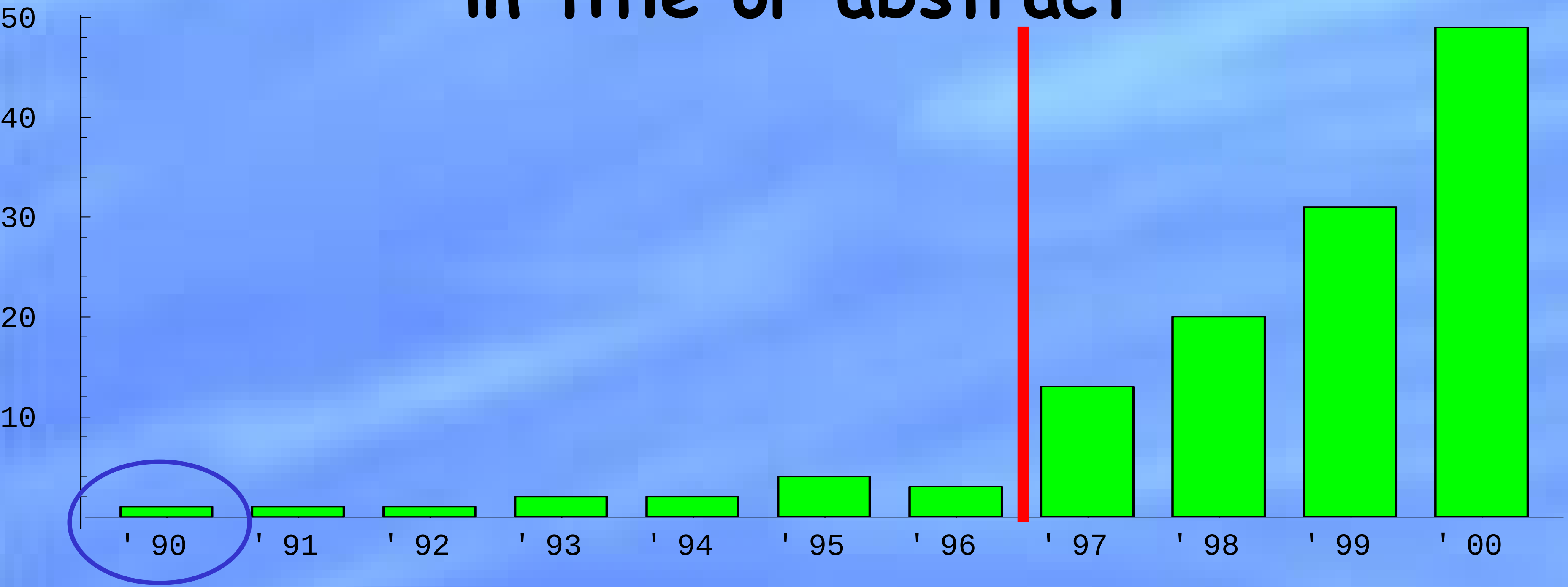
What is being plotted?



Answer: Number of papers with "quantum entanglement" in title or abstract



N. D. Mermin, Phys. Rev. Lett. (1990)

Entanglement is a *physical resource*:
Bennett, DiVincenzo, Smolin and Wootters,
Phys. Rev. A (November, 1996)

Entanglement

Michael A. Nielsen

University of Queensland

Goals:

1. To explain why we regard entanglement as a physical resource, like energy or mass.
2. To explain how entanglement can be quantified.
3. To explain how the quantitative theory of entanglement can be used to gain insight into quantum information processing, and into other physical processes.

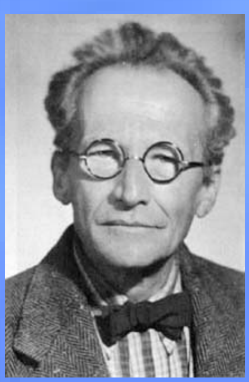
Entanglement revisited

Alice

Bob

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$|\psi\rangle \neq |a\rangle|b\rangle$$



Schroedinger (1935): "I would not call [entanglement] *one* but rather *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought."

Entanglement and classicality

Bell (1964) and Aspect (1982): Entanglement can be used to show that no "locally realistic" (that is, classical) theory of the world is possible.

Further reading: Asher Peres, "Quantum theory: concepts and methods", Kluwer (1993).

Using entanglement to do stuff

superdense coding

entanglement-based quantum cryptography

quantum teleportation

quantum computing

Entanglement is a useful resource that can be used to accomplish tasks that would otherwise be difficult or impossible.

Given an information-processing goal, we can always ask "What would I gain by throwing some entanglement into the problem?"

Representation independence of entanglement

Properties independent of physical representation

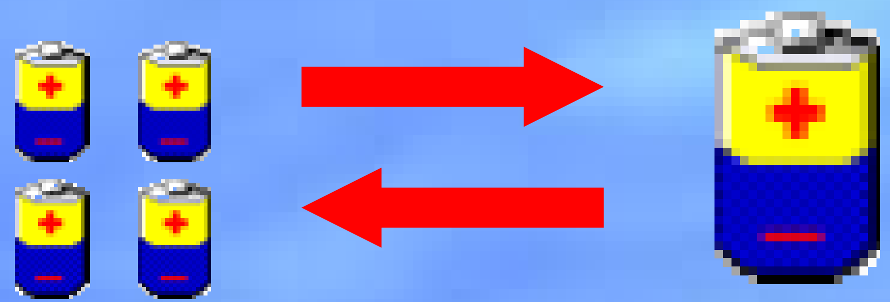
$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Electron spin: $\frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}$

Photon polarization: $\frac{|HH\rangle + |VV\rangle}{\sqrt{2}}$

etcetera

Qualitative equivalence of different entangled states



2 copies of $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|11\rangle + \frac{1}{2}|22\rangle$ is equivalent to

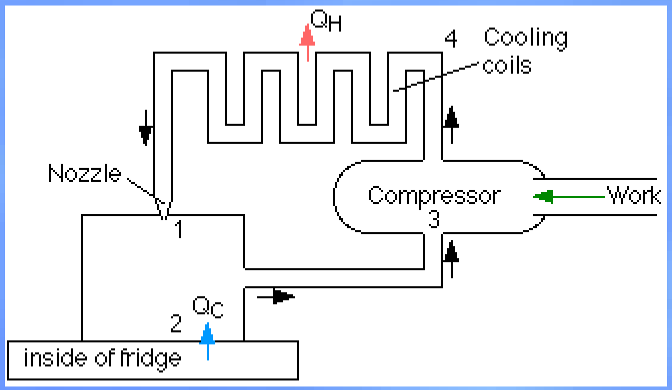
3 copies of $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$!

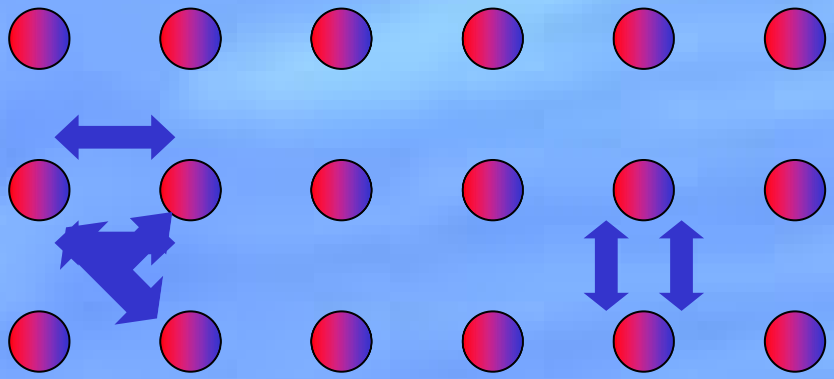
Summary

1. Entanglement is not classical.
2. Entanglement is a resource that can be used to do interesting things.
3. Entanglement has properties independent of physical representation.
4. Different entangled states are qualitatively equivalent to one another.

Can we develop a quantitative theory of entanglement?

What might we get out of such a theory?



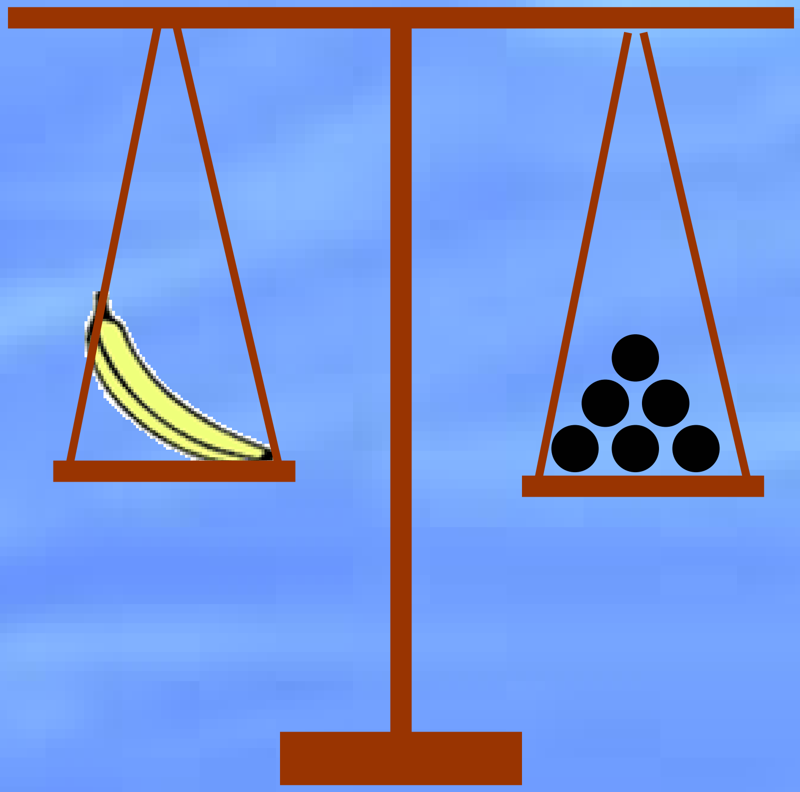


Thermodynamics is a set of high-level principles governing the behaviour of energy.

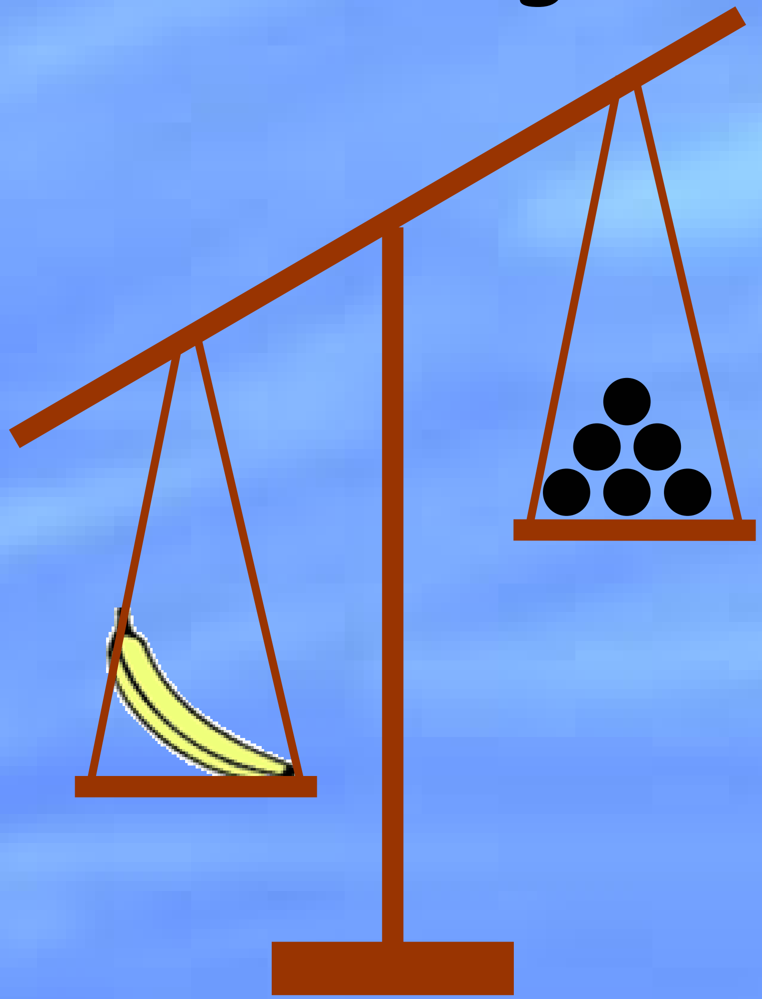
We hope that the theory of entanglement will be a similarly powerful set of high-level principles governing entanglement.

(Figure taken from Boston University's 1999 PY105 class.)

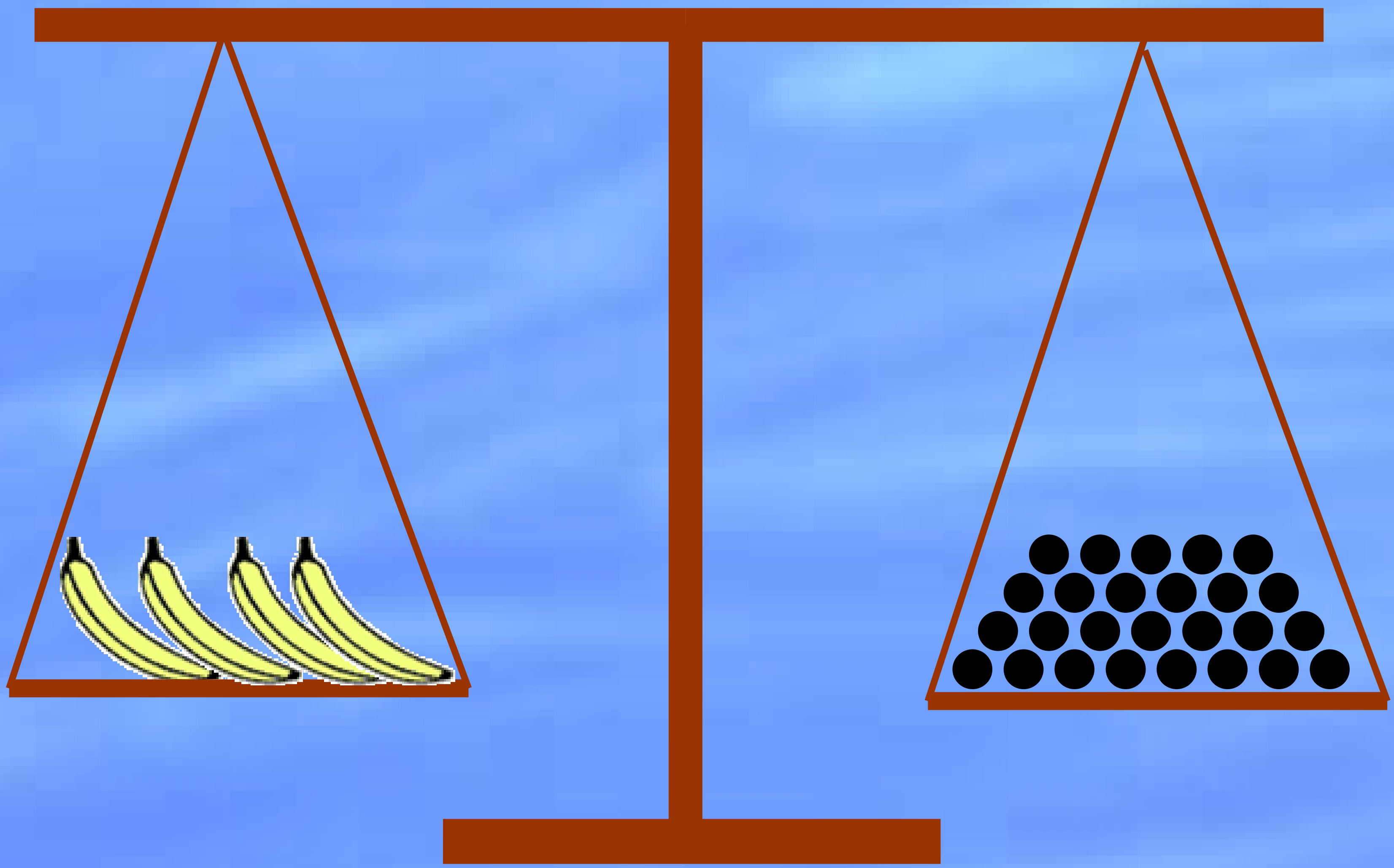
How massive is a given object?



How massive is a given object?



How massive is a given object?



How massive is a given object?

$$\text{Mass} \equiv \lim \frac{\text{number of standard masses}}{\text{number of copies of object}}$$

A standard unit for entanglement

Alice

Bob

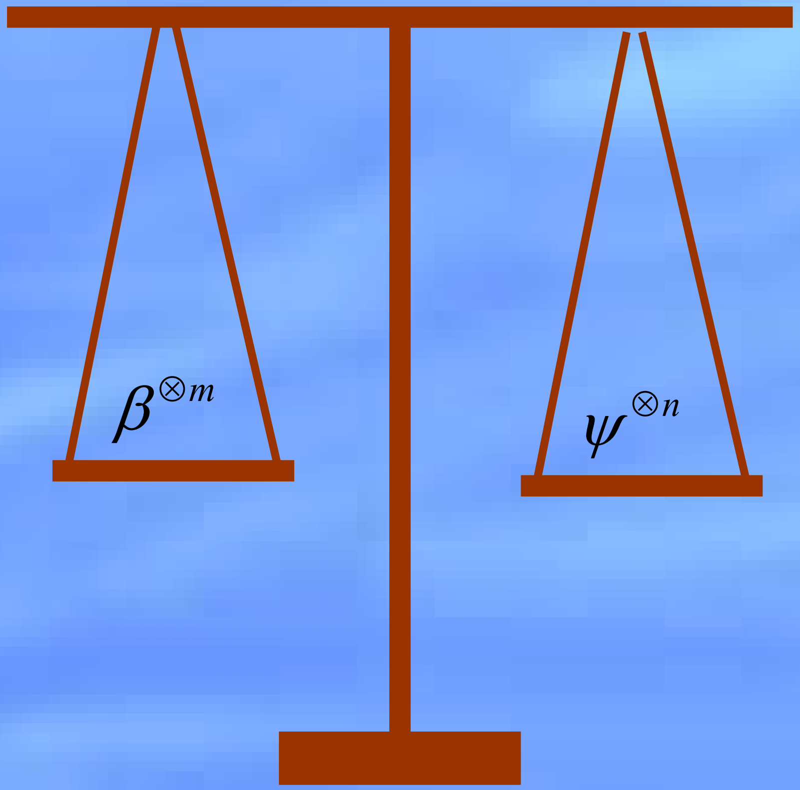
$$|\beta\rangle \equiv \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Question: Why use the Bell state as the standard unit?

Answer: "Because it's there" – we'll do so because it's clearly an important state, and in the spirit of exploration.

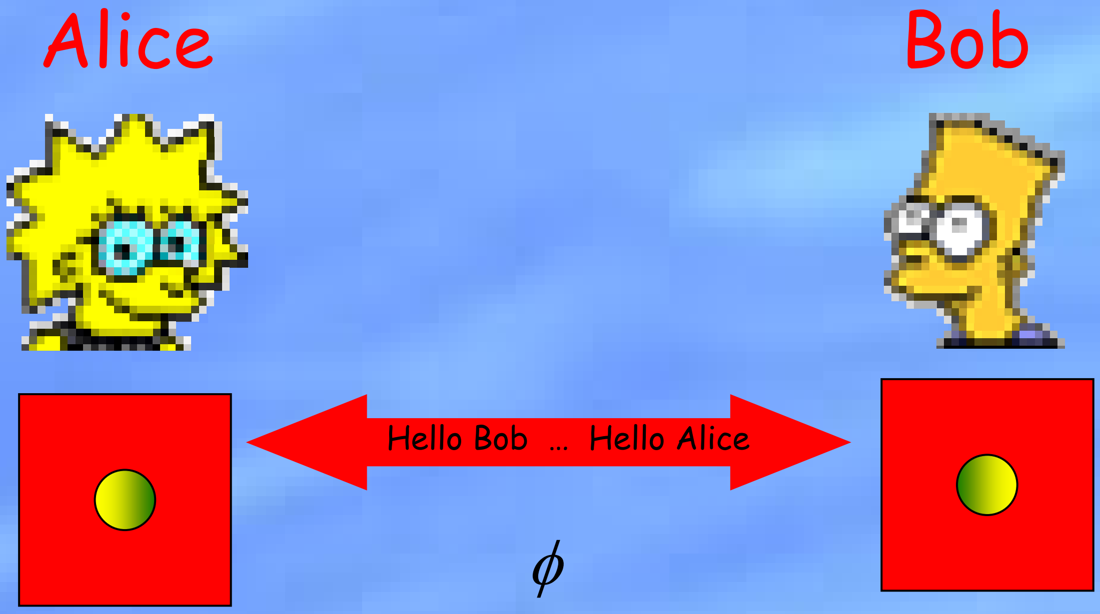
Answer: Later on, we'll see that choosing the Bell state leads to some interesting connections with other problems.

How can we "balance" entanglement?

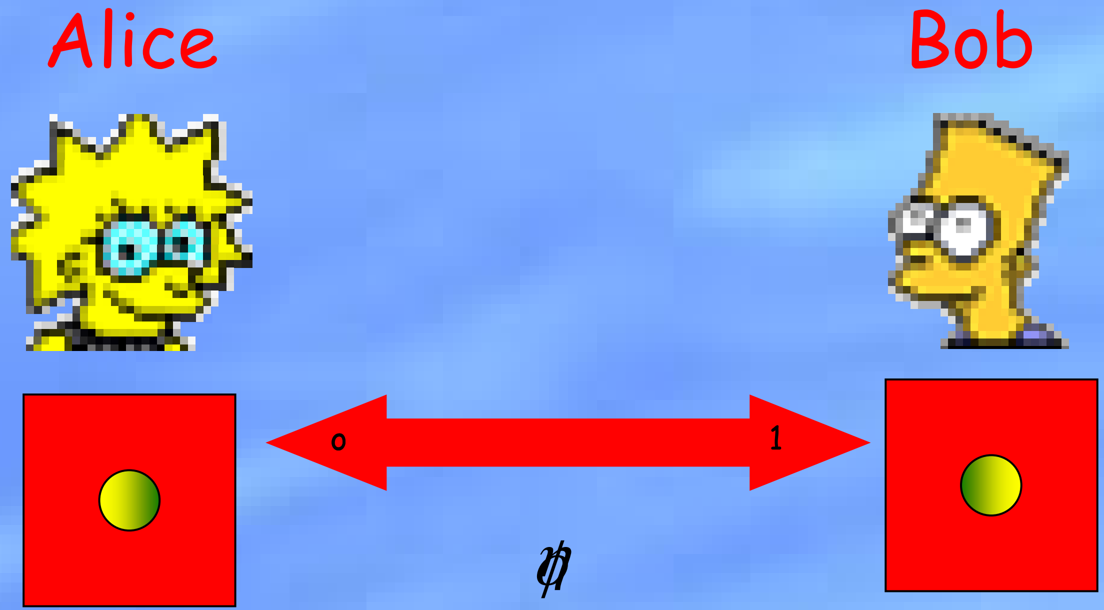


$$\text{Entanglement} \equiv \lim \frac{m}{n}$$

What it means for one state to be "at least as entangled" as another



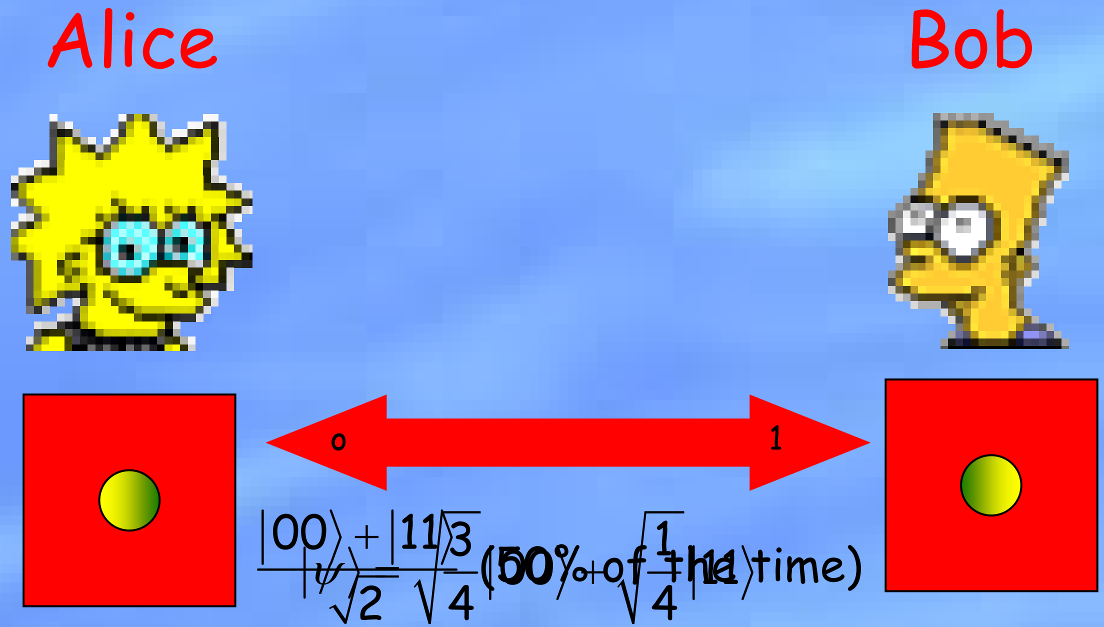
What it means for one state to be "at least as entangled" as another



$$\text{Entanglement}(\phi) \geq \text{Entanglement}(\eta)$$

ϕ can be converted to η by an LOCC ("local operations and classical communication") protocol.

An example of an LOCC protocol



Such a protocol will let us distill n copies of

$$|\psi\rangle = \sqrt{\frac{3}{4}}|00\rangle + \sqrt{\frac{1}{4}}|11\rangle \text{ into } \approx \frac{n}{2} \text{ Bell pairs.}$$

How the protocol works

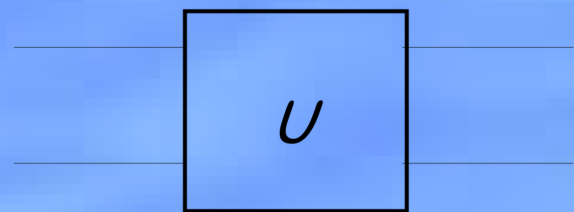
$$|\psi\rangle = \sqrt{\frac{3}{4}}|00\rangle + \sqrt{\frac{1}{4}}|11\rangle$$

$$U|00\rangle = \tfrac{1}{\sqrt{3}}|00\rangle + \sqrt{\tfrac{2}{3}}|11\rangle$$
$$U|01\rangle = |01\rangle$$

Consider the circuit

$|0\rangle$

$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$



measure m

$|\phi_{m'}\rangle$

Exercise: Find a circuit of controlled-nots and single-qubit unitaries to implement U.

Exercise: Show that

$$\Pr(0) = 1 - \frac{2|\alpha|^2}{3} \text{ and } |\phi_0\rangle \propto \frac{\alpha}{\sqrt{3}}|0\rangle + \beta|1\rangle.$$

How the protocol works

$$|\psi\rangle = \sqrt{\frac{3}{4}}|00\rangle + \sqrt{\frac{1}{4}}|11\rangle$$

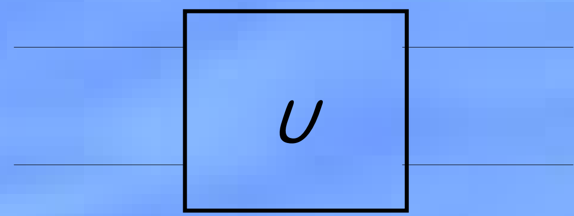
$$U|00\rangle = \tfrac{1}{\sqrt{3}}|00\rangle + \sqrt{\tfrac{2}{3}}|11\rangle$$

$$U|01\rangle = |01\rangle$$

Consider the circuit

$|0\rangle$

$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$



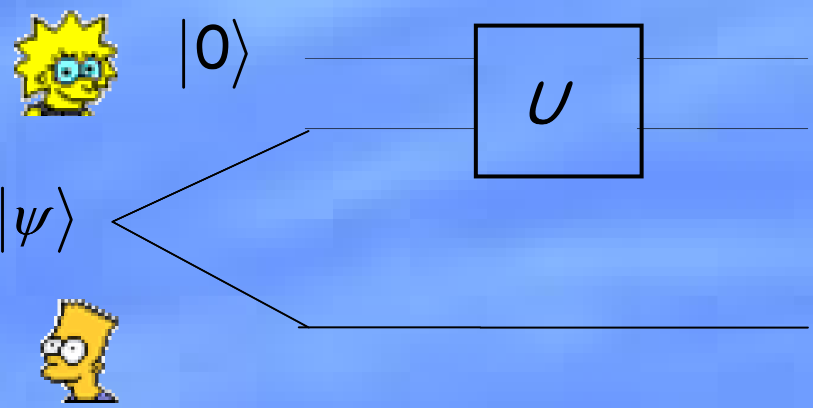
measure m

$|\phi_{m'}\rangle$

Distillation procedure:

$|0\rangle$

$|\psi\rangle$



measure 0

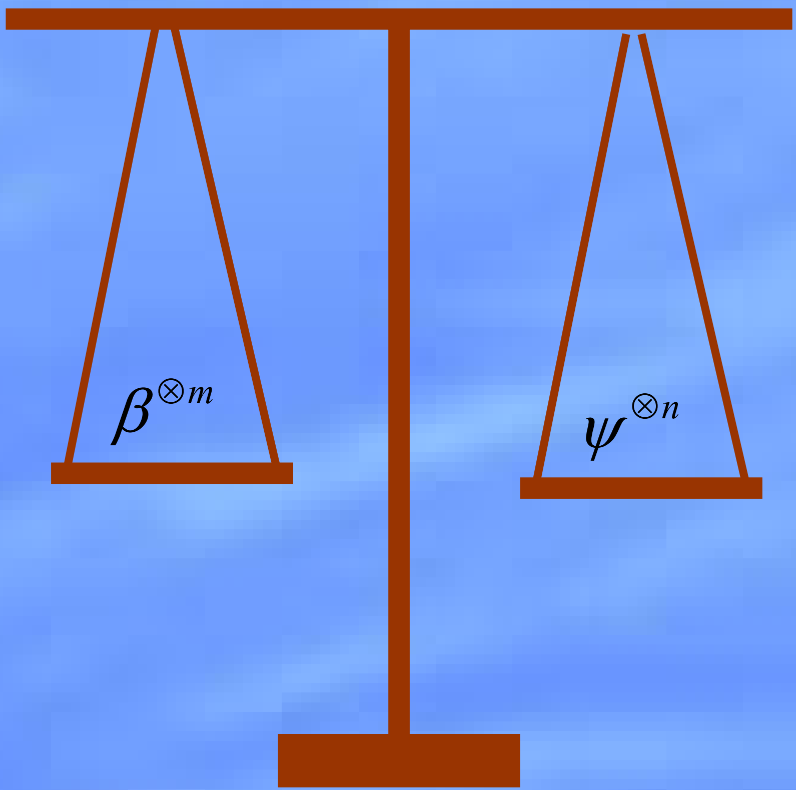
$$\sqrt{\frac{3}{4}}|00\rangle + \sqrt{\frac{1}{4}}|11\rangle \rightarrow$$

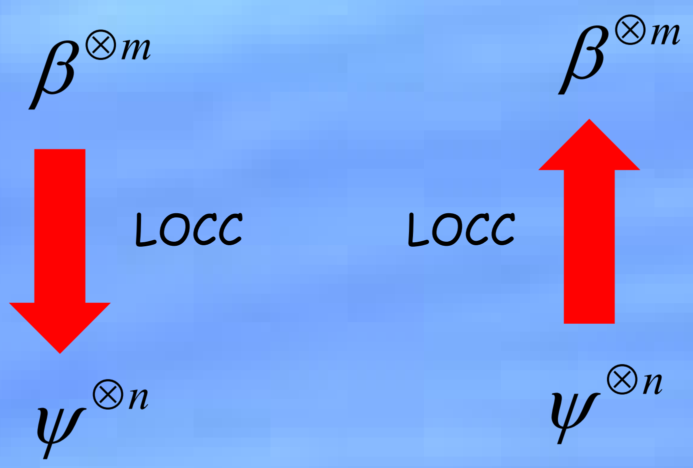
$$|\psi_0'\rangle \propto \frac{\sqrt{3/4}}{\sqrt{3}}|00\rangle + \sqrt{\frac{1}{4}}|11\rangle$$

$$= \frac{|00\rangle + |11\rangle}{\sqrt{2}} \text{ w. p. } \frac{1}{2}$$

Thus n copies of $|\psi\rangle \rightarrow \frac{n}{2}$ Bell pairs

Back to balancing entanglement





Not possible in general!

$$\text{Entanglement} \equiv \lim \frac{m}{n}$$

How to balance entanglement

For any $\varepsilon > 0$ and sufficiently large m and n:

$\beta^{\otimes m}$ ↓ $\psi^{\otimes n}$

$\beta^{\otimes m(1-\varepsilon)}$ ↑ $\psi^{\otimes n}$

$$\text{Entanglement} \equiv \lim \frac{m}{n}$$

$$n \times \psi \leftrightarrow n \times \mathrm{E}(\psi) \text{ Bell states}$$

E(ψ) is the maximal number of Bell states that can be distilled, per copy of ψ.

$n \times \mathrm{E}(\psi)$ Bell states

↓

$n \times \psi$

↓

$n \times k$ Bell states

Exercise: Show that by local operations and classical communication, Alice and Bob can't increase the number of Bell pairs they share.

How much entanglement?

Alice

Bob

$$|\psi\rangle$$

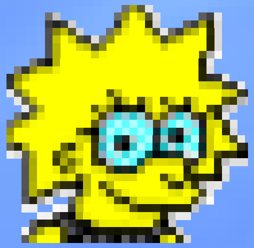
$$\rho_A = \mathrm{tr}_B\left(|\psi\rangle\langle\psi|\right) \qquad \rho_B = \mathrm{tr}_A\left(|\psi\rangle\langle\psi|\right)$$

$$E(\psi) = S(\rho_A) = S(\rho_B)$$

That is, $n \times \psi \leftrightarrow nS(\rho_A)$ Bell states.

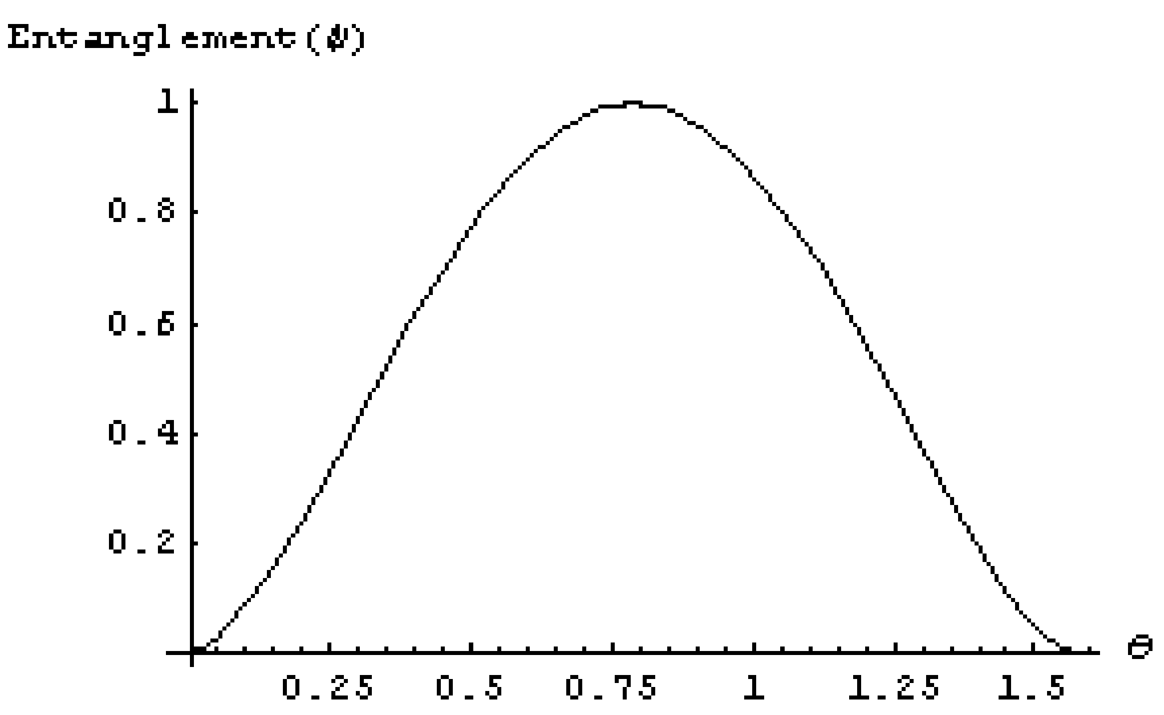
Example

Alice



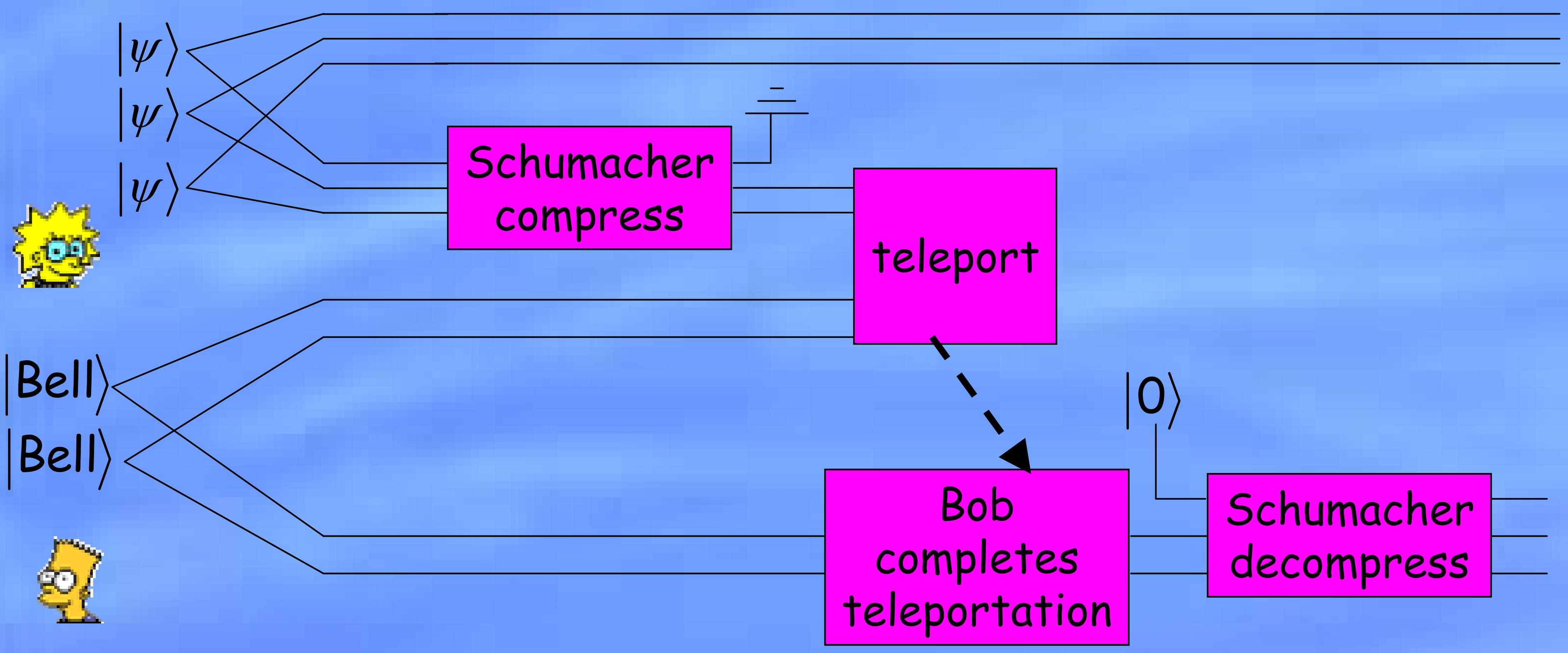
Bob

$$|\psi\rangle = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle$$



How to go from $nS(\rho_A)$ Bell states to n copies of ψ, by LOCC

Suppose $S(\rho_A) = \frac{2}{3}$.



An entangled analogue to the second law of thermodynamics

Entanglement can only *decrease* under local operations and classical communication

ψ $\rightarrow$ ϕ

$n \times \psi$ $\rightarrow$ $n \times \phi$ $\rightarrow$ $n \times$ E(ϕ) Bell states

E(ψ) is the maximal number of Bell states that can be distilled, per copy of ψ.

$$E(\psi) \geq E(\phi)$$

Approximate teleportation

Alice

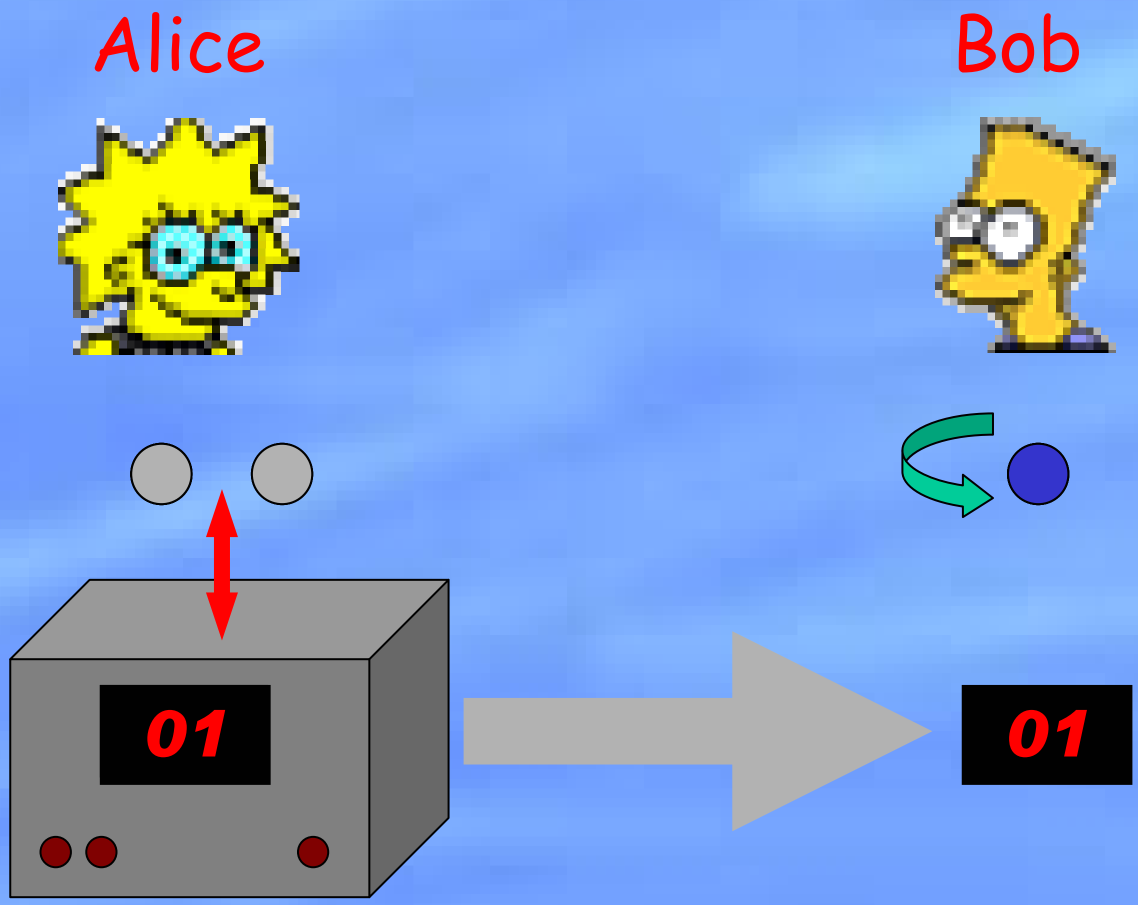
Bob

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Approximate teleportation

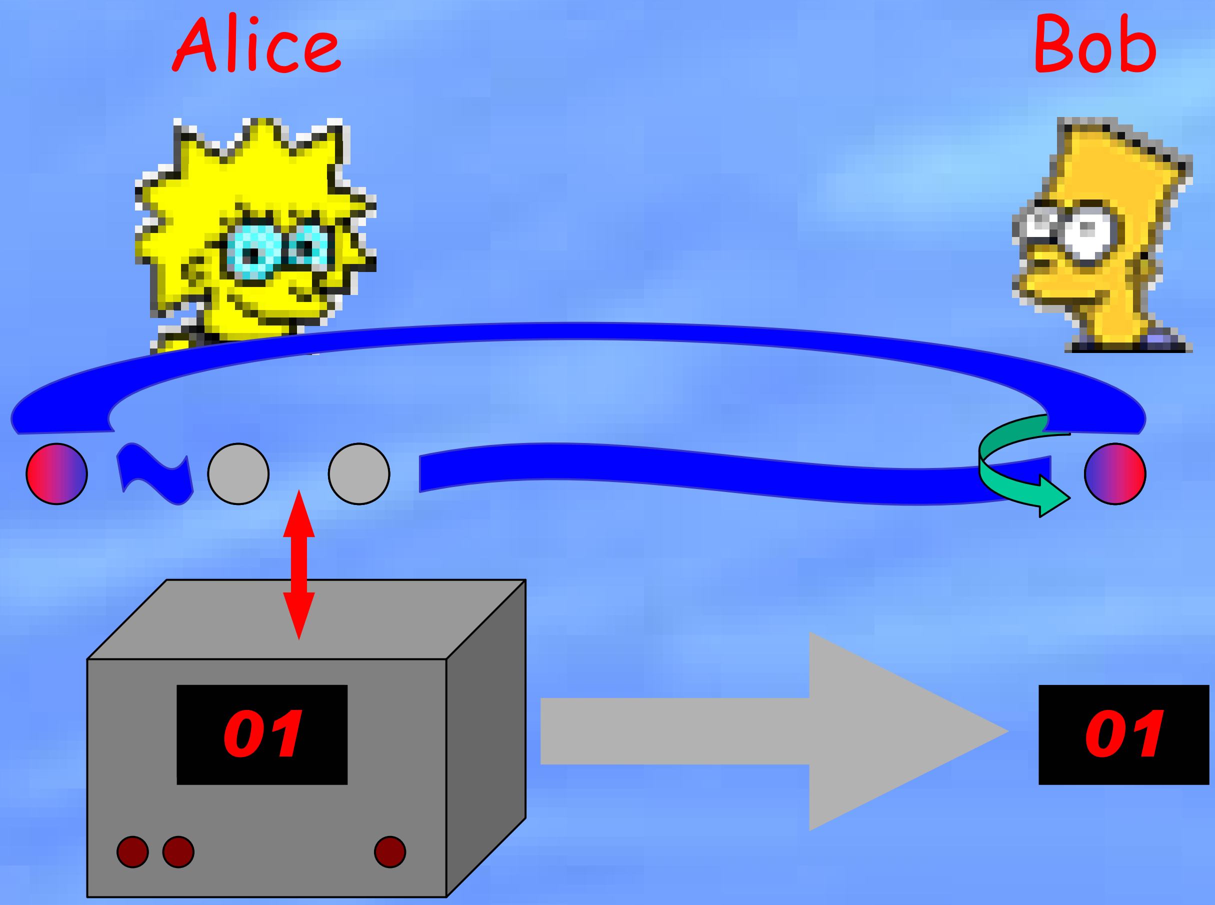
Alice Bob

The original teleportation protocol



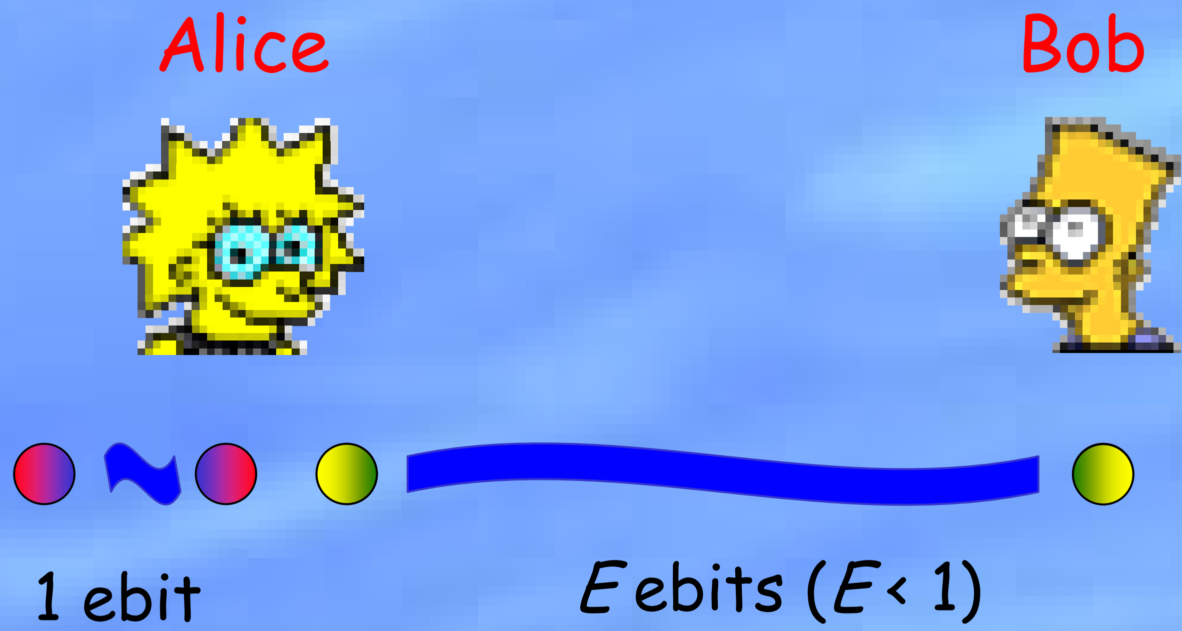
Teleporting entanglement

Teleporting entanglement



The ability to teleport an arbitrary state implies the ability to teleport entanglement

Approximate teleportation



Total initial entanglement between Alice and Bob at most E ebits.

If Alice and Bob only do local operations and classical communication then the final entanglement between their systems cannot be more than when it started.

Approximate teleportation

Since the final entanglement is not 1 ebit, some states must be imperfectly teleported.

Approximate teleportation

Alice

Bob

ψ $\quad$ E ebits (E < 1) $\quad$ ρ

$$F_{\min} = \min_{\psi} \langle \psi | \rho | \psi \rangle$$

$$F_{\min} \leq 1 - \frac{1}{3}(1 - E)$$

Back to the "Why Bell states?" question

Teleportation: shared entanglement and classical communication enables the communication of qubits.

Alice

$|\psi\rangle$

$|\psi\rangle$

$|\psi\rangle$

Bob

Physical resource: Alice and Bob share a large number of copies of $|\psi\rangle$, and can do unlimited classical communication, as well as arbitrary operations on their local systems.

Information processing task: Alice wants to send qubits to Bob.

Criterion for success: The qubit communication should take place with fidelity approaching one.

How many copies of $|\psi\rangle$ are needed to reliably communicate a qubit from Alice to Bob?

$$\text{Entanglement}(\psi) \equiv \frac{\text{max \# of qubits that can be communicated}}{\text{copy of } |\psi\rangle}$$