Quantum Noise

Michael A. Nielsen

University of Queensland

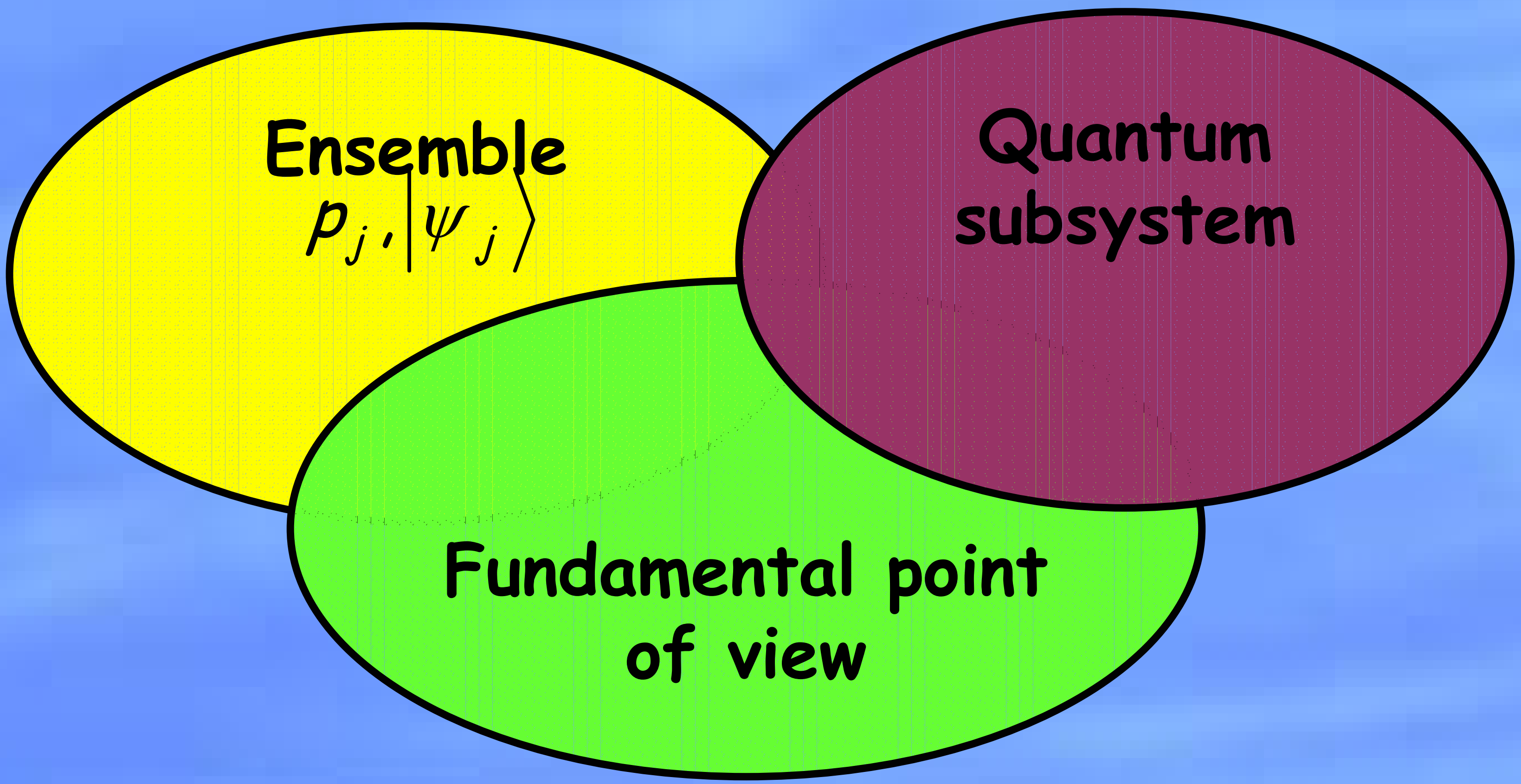
Goals:

1. To introduce a tool – the density matrix – that is used to describe noise in quantum systems, and to give some examples.

Density matrices

Generalization of the quantum state used to describe noisy quantum systems.

Terminology: "Density matrix" = "Density operator"



What we're going to do in this lecture, and why we're doing it

Most of the lecture will be spent understanding the density matrix.

Unfortunately, that means we've got to master a rather complex formalism.

It might seem a little strange, since the density matrix is *never* essential for calculations – it's a mathematical tool, introduced for convenience.

Why bother with it?

The density matrix seems to be a very deep abstraction – once you've mastered the formalism, it becomes far easier to understand many other things, including quantum noise, quantum error-correction, quantum entanglement, and quantum communication.

I. Ensemble point of view

Imagine that a quantum system is in the state $|\psi_j\rangle$ with probability p_j.

We do a measurement described by projectors P_k.

$$\text{Probability of outcome } k = \sum_k \Pr\left(k \,|\, \text{state } \psi_j\right) p_j$$
$$= \sum_k \langle \psi_j | P_k | \psi_j \rangle p_j$$
$$= \sum_k p_j \operatorname{tr}\left(|\psi_j\rangle\langle\psi_j| P_k\right)$$

$$\text{Probability of outcome } k = \operatorname{tr}(\rho P_k)$$

where $\rho \equiv \sum_j p_j |\psi_j\rangle\langle\psi_j|$ is the density matrix.

ρ completely determines all measurement statistics.

Qubit examples

Suppose $|\psi\rangle = |0\rangle$ with probability 1.

Then $\rho = |0\rangle\langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$

Suppose $|\psi\rangle = |1\rangle$ with probability 1.

Then $\rho = |1\rangle\langle 1| = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$

Suppose $|\psi\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$ with probability 1.

Then $\rho = \left(\frac{|0\rangle + i|1\rangle}{\sqrt{2}} \right) \left(\frac{\langle 0| - i\langle 1|}{\sqrt{2}} \right) = \frac{1}{2} \begin{bmatrix} 1 \\ i \end{bmatrix} \begin{bmatrix} 1 & -i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}.$

Qubit example

Suppose $|\psi\rangle = |0\rangle$ with probability p, and $|\psi\rangle = |1\rangle$ with probability $1-p$.

$$\text{Then } \rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$$
$$= p\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (1-p)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}.$$

Measurement in the $|0\rangle, |1\rangle$ basis yields

$$\Pr(0) = \mathrm{tr}\left(\rho|0\rangle\langle 0|\right) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= p.$$

Similarly, $\Pr(1) = 1-p$.

Why work with density matrices?

Answer: Simplicity!

The quantum state is:

$|0\rangle$ with probability 0.1

$|1\rangle$ with probability 0.1

$\frac{|0\rangle + |1\rangle}{\sqrt{2}}$ with probability 0.15

$\frac{|0\rangle - |1\rangle}{\sqrt{2}}$ with probability 0.15

$\frac{|0\rangle + i|1\rangle}{\sqrt{2}}$ with probability 0.25

$\frac{|0\rangle - i|1\rangle}{\sqrt{2}}$ with probability 0.25

$$\rho = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Two-qubit example

Suppose $|\psi\rangle = |00\rangle$ with probability p, and $|\psi\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$ with probability $1-p$. Then:

$$\rho = p|00\rangle\langle 00| + (1-p) \times \frac{1}{2}\left(|01\rangle + |10\rangle\right)\left(\langle 01| + \langle 10|\right)$$

$$= p\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} + \frac{1-p}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & \frac{1-p}{2} & \frac{1-p}{2} & 0 \\ 0 & \frac{1-p}{2} & \frac{1-p}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Dynamics and the density matrix

Suppose we have a quantum system in the state $|\psi_j\rangle$ with probability p_j.

The quantum system undergoes a dynamics described by the unitary matrix U.

The quantum system is now in the state $U|\psi_j\rangle$ with probability p_j.

The initial density matrix is $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$.

The final density matrix is $\rho' = \sum_j p_j U|\psi_j\rangle\langle\psi_j|U^\dagger$.

$$= U\left(\sum_j p_j U|\psi_j\rangle\langle\psi_j|\right)U^\dagger.$$

$$\rho' = U\rho U^\dagger.$$

Single-qubit examples

Suppose $|\psi\rangle = |0\rangle$ with probability p, and $|\psi\rangle = |1\rangle$ with probability $1-p$.

Then $\rho = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}$.

Suppose an X gate is applied. Then $\rho' = X\rho X = \begin{bmatrix} 1-p & 0 \\ 0 & p \end{bmatrix}$.

Suppose $|\psi\rangle = |0\rangle$ and $|\psi\rangle = |1\rangle$ with equal probabilities $\frac{1}{2}$.

Then $\rho = \dfrac{I}{2}$. ⟵ "Completely mixed state"

Suppose any unitary gate U is applied.

Then $\rho' = U\dfrac{I}{2}U^{\dagger} = \dfrac{I}{2}$.

How the density matrix changes during a measurement

Worked Exercise : Suppose a measurement described by projectors P_k is performed on an ensemble giving rise to the density matrix ρ. If the measurement gives result k show that the corresponding post-measurement density matrix is

$$\rho_k^{'} = \frac{P_k \rho P_k}{\mathrm{tr}\left(P_k \rho P_k\right)}.$$

Characterizing the density matrix

What class of matrices correspond to possible density matrices?

Suppose $\rho = \sum_j p_j \left|\psi_j\right\rangle\left\langle\psi_j\right|$ is a density matrix.

Then $\mathrm{tr}(\rho) = \sum_j p_j \mathrm{tr}\left(\left|\psi_j\right\rangle\left\langle\psi_j\right|\right) \; = \sum_j p_j = 1$

For any vector $|a\rangle$,

$$\langle a|\rho|a\rangle = \sum_j p_j \langle a\left|\psi_j\right\rangle\left\langle\psi_j\right|a\rangle \;\; = \sum_j p_j \left|\left\langle a\middle|\psi_j\right\rangle\right|^2 \geq 0$$

Summary: $\mathrm{tr}(\rho)$=1 and ρ is a positive matrix.

Exercise: Given that $\mathrm{tr}(\rho)$=1 and ρ is a positive matrix, show that there is some set of states $\left|\psi_j\right\rangle$ and probabilities p_j such that $\rho = \sum_j p_j \left|\psi_j\right\rangle\left\langle\psi_j\right|$.

Summary of the ensemble point of view

Definition: The density matrix for a system in state $|\psi_j\rangle$ with probability p_j is $\rho \equiv \sum_j p_j |\psi_j\rangle\langle\psi_j|$.

Dynamics: $\rho \to \rho' = U \rho U^\dagger$.

Measurement: A measurement described by projectors P_k gives result k with probability $\mathrm{tr}(P_k \rho)$, and the post-measurement density matrix is $\rho_k' = \frac{P_k \rho P_k}{\mathrm{tr}(P_k \rho P_k)}$.

Characterization: $\mathrm{tr}(\rho)=1$, and ρ is a positive matrix. Conversely, given any matrix satisfying these properties, there exists a set of states $|\psi_j\rangle$ and probabilities p_j such that $\rho=\sum_j p_j |\psi_j\rangle\langle\psi_j|$.

A simple example of quantum noise

With probability p the not gate is applied.

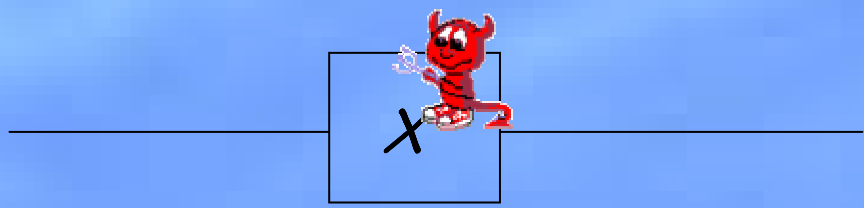
With probability 1-p the not gate fails, and nothing happens.

$$\rho = \sum_j p_j \left|\psi_j\right\rangle\left\langle\psi_j\right| \rightarrow \sum_j \left(p_j p X \left|\psi_j\right\rangle\left\langle\psi_j\right| X + p_j (1-p) \left|\psi_j\right\rangle\left\langle\psi_j\right| \right)$$

$$= pX\rho X + (1-p)\rho$$

If we were to work with state vectors instead of density matrices, doing a series of noisy quantum gates would quickly result in an incredibly complex ensemble of states.

How good a not gate is this?



How "good" a not gate is this, for a particular input $|\psi\rangle$?

$$\rho \to E(\rho) \equiv pX\rho X + (1-p)\rho$$

A quantum operation

We compare the ideal output, $X|\psi\rangle$, to the actual output.

The usual way two states $|a\rangle$ and $|b\rangle$ are compared is to compute the fidelity, or overlap:

$$F(a,b) \equiv |\langle a|b\rangle|.$$

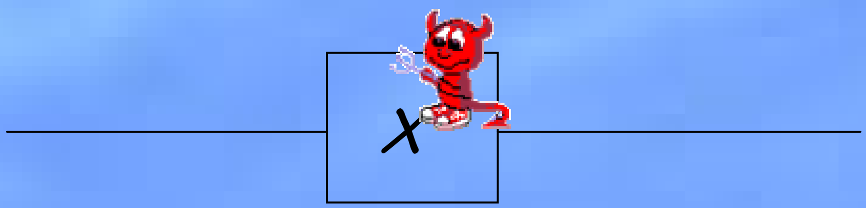
The fidelity measures how similar the states are, ranging from 0 (totally dissimilar), up to 1 (the same).

To compare $|a\rangle$ with $\sigma = \sum_j p_j |\phi_j\rangle\langle\phi_j|$ we compute the fidelity,

$$F(a,\sigma) \equiv \sqrt{\langle a|\sigma|a\rangle}.$$

Fidelity measures for two mixed states are a surprisingly complex topic!

How good a not gate is this?



How "good" a not gate is this, for a particular input $|\psi\rangle$?

$$\rho \rightarrow E(\rho) \equiv pX\rho X + (1-p)\rho$$

We compare the ideal output, $X|\psi\rangle$, to the actual output.

The fidelity of the gate is thus

$$F(X|\psi\rangle, E(|\psi\rangle\langle\psi|)) = \sqrt{\langle\psi|XE(|\psi\rangle\langle\psi|)X|\psi\rangle}$$

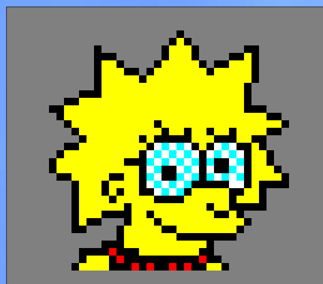
$$= \sqrt{p\langle\psi|\psi\rangle\langle\psi|\psi\rangle + (1-p)\langle\psi|X|\psi\rangle\langle\psi|X|\psi\rangle}$$

$$= \sqrt{p + (1-p)\langle\psi|X|\psi\rangle^2}$$

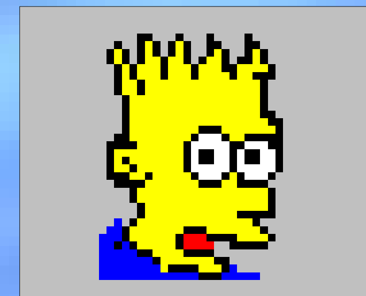
The fidelity ranges between $\sqrt{p}$, for $|\psi\rangle = |0\rangle$, and 1, for $|\psi\rangle = \left(|0\rangle + |1\rangle / \sqrt{2}\right)$.

II. Subsystem point of view

Alice



Bob



$$P_j \qquad |\psi\rangle = \sum_{kl} \alpha_{kl} |k\rangle |l\rangle$$

$$\Pr(j) = \mathrm{tr}\left(\left(P_j \otimes I\right)|\psi\rangle\langle\psi|\right) = \sum_{klmn} \alpha_{kl}\alpha^*_{mn} \mathrm{tr}\left(\left(P_j \otimes I\right)|k\rangle\langle l| \otimes |m\rangle\langle n|\right)$$

$$= \sum_{klmn} \alpha_{kl}\alpha^*_{mn} \langle l|\langle n|\left(P_j \otimes I\right)|k\rangle|m\rangle \;\; = \sum_{klm} \alpha_{kl}\alpha^*_{ml} \langle l|P_j|k\rangle$$

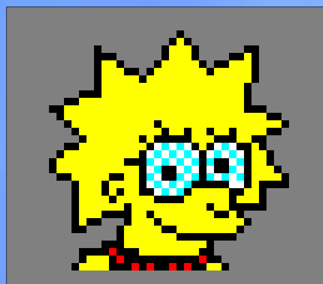
$$= \sum_{klm} \alpha_{kl}\alpha^*_{ml} \mathrm{tr}\left(P_j |k\rangle\langle l|\right)$$

$$= \mathrm{tr}\left(P_j \rho_A\right)$$

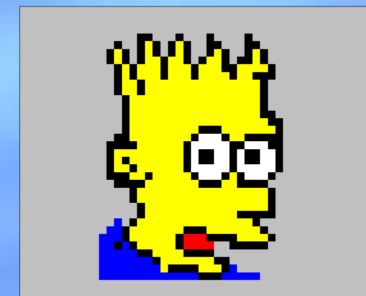
where $\rho_A \equiv \sum_{klm} \alpha_{kl}\alpha^*_{ml} |k\rangle\langle l|$ is known as the reduced density matrix of system A.

II. Subsystem point of view

Alice



Bob



P_j $\quad |\psi\rangle = \sum_{kl} \alpha_{kl} |k\rangle |l\rangle$

$$\Pr(j) = \mathrm{tr}\left(P_j \rho_A\right), \text{ where}$$
$$\rho_A \equiv \mathrm{tr}_B\left(|\psi\rangle\langle\psi|\right) \equiv \sum_{klm} \alpha_{kl} \alpha^*_{ml} |k\rangle\langle l|$$

ρ_A is the reduced density matrix for system *A*.

All the statistics for measurements on system *A* can be recovered from ρ_A.

How to calculate: a method, and an example

An alternative, more convenient definition for the partial trace is to define:

$$\mathrm{tr}_B\left(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|\right) \equiv |a_1\rangle\langle a_2| \times \mathrm{tr}\left(|b_1\rangle\langle b_2|\right)$$
$$= \langle b_2|b_1\rangle |a_1\rangle\langle a_2|$$

Then extend the definition linearly to arbitrary matrices.

Exercise: Show that this new definition agrees with the old, that is, $\mathrm{tr}_B\left(|\psi\rangle\langle\psi|\right) = \sum_{klm} \alpha_{kl}\alpha^*_{ml} |k\rangle\langle l|$ when $|\psi\rangle = \sum_{kl} \alpha_{kl} |k\rangle|l\rangle$.

Example: If the system is in the state $|a\rangle|b\rangle$ then

$$\rho_A = \mathrm{tr}_B\left(|a\rangle\langle a| \otimes |b\rangle\langle b|\right) = \langle b|b\rangle |a\rangle\langle a| = |a\rangle\langle a|$$

The example of a Bell state

Example: Suppose $|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$. Then the reduced density matrix for the first system is given by:

$$\rho_A = \mathrm{tr}_B\left(|\psi\rangle\langle\psi|\right)$$

$$= \frac{\mathrm{tr}_B\left(|00\rangle\langle 00|\right) + \mathrm{tr}_B\left(|00\rangle\langle 11|\right) + \mathrm{tr}_B\left(|11\rangle\langle 00|\right) + \mathrm{tr}_B\left(|11\rangle\langle 11|\right)}{2}$$

$$= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}$$

$$= \frac{I}{2}$$

From Alice's point of view, it's just like having the state $|0\rangle$ with probability $\frac{1}{2}$, and the state $|1\rangle$ with probability $\frac{1}{2}$.

Under dynamics and measurement, the density matrix behaves just as it does in the ensemble point of view.

III. The density matrix as fundamental object

Postulate 1: A quantum system is described by a positive matrix (the *density matrix*), with unit trace, acting on a complex inner product space known as *state space*.

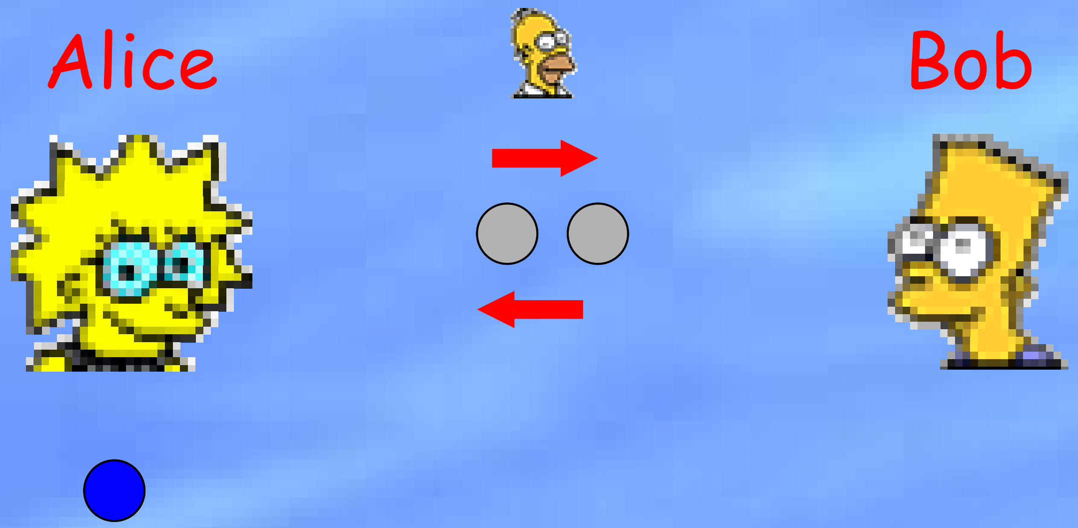
A system in state ρ_j with probability p_j has density matrix $\sum_j p_j \rho_j$.

Postulate 2: The dynamics of a closed quantum system are described by $\rho \to \rho' = U \rho U^\dagger$.

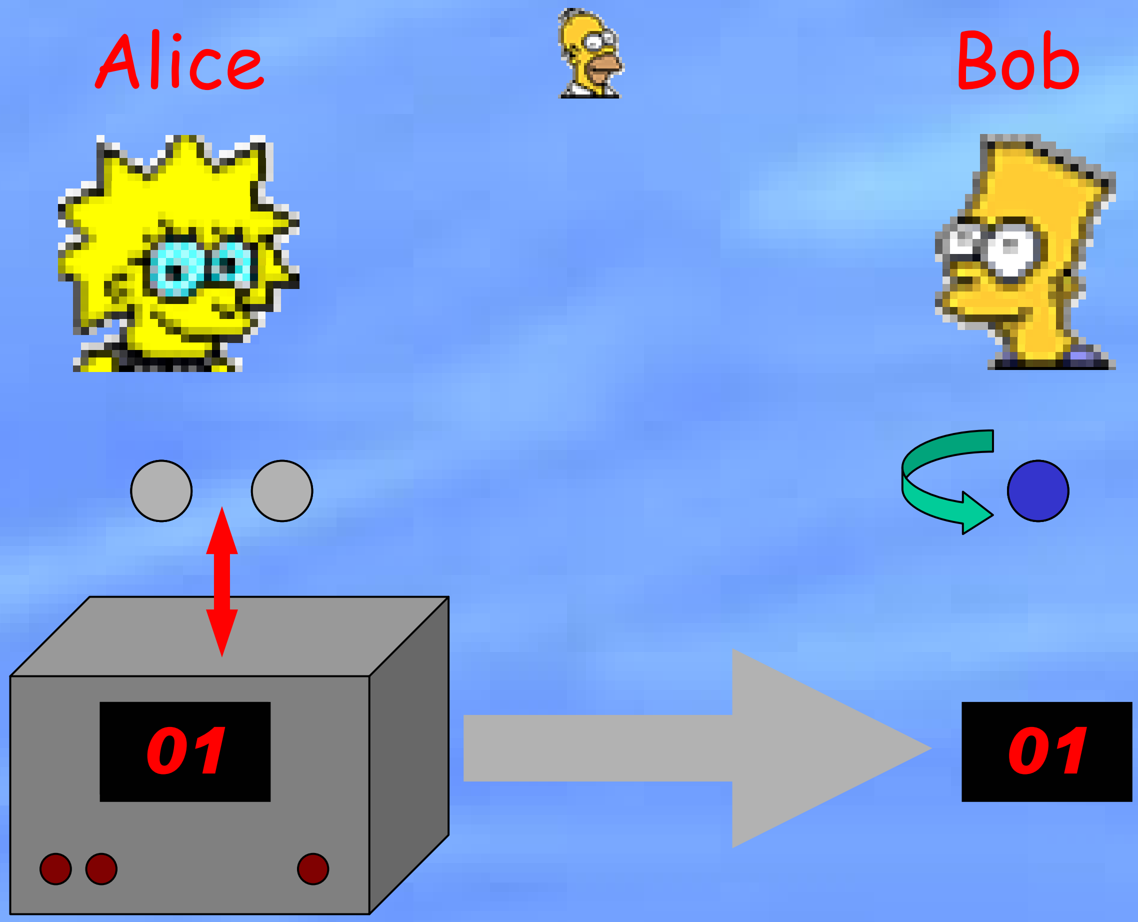
Postulate 3: A measurement described by projectors P_k gives result k with probability $\mathrm{tr}(P_k \rho)$, and the post-measurement density matrix is $\rho_k' = \frac{P_k \rho P_k}{\mathrm{tr}(P_k \rho P_k)}$.

Postulate 4: We take the tensor product to find the state space of a composite system. The state of one component is found by taking the partial trace over the remainder of the system.

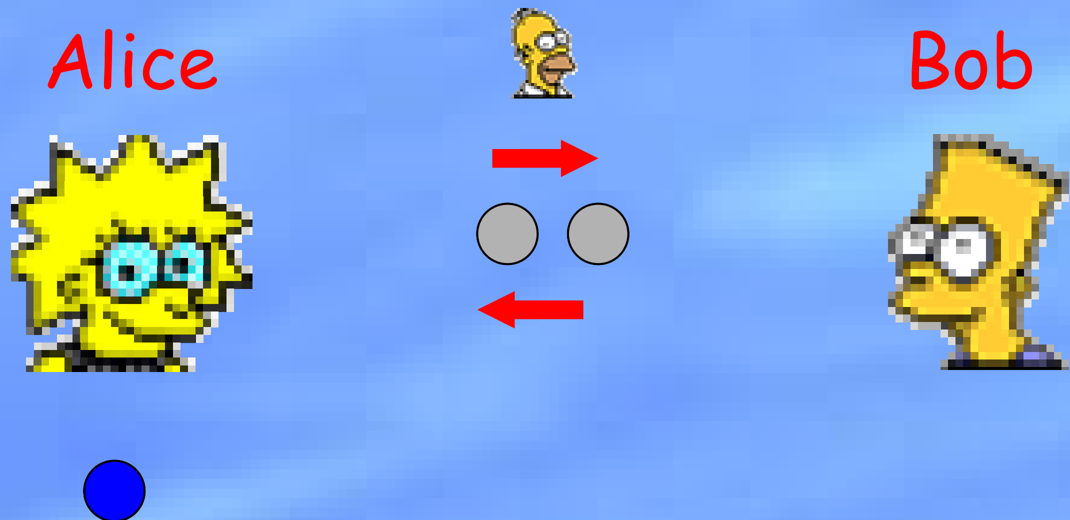
Why teleportation doesn't allow FTL communication



Why teleportation doesn't allow FTL communication



Why teleportation doesn't allow FTL communication



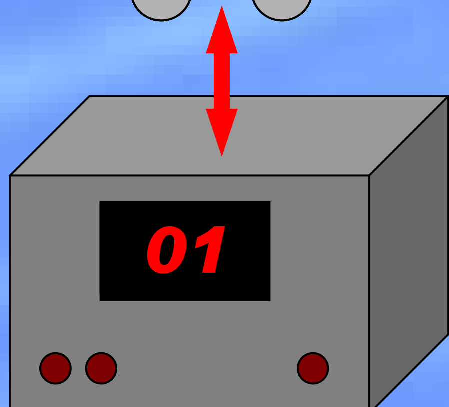
The initial state for the protocol is $|\psi\rangle\left(\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right)$

Bob's initial reduced density matrix is just the reduced density matrix for a Bell state, $\rho_B = \frac{I}{2}$.

Why teleportation doesn't allow FTL communication

Alice

Bob



$$\frac{|B_1\rangle|\psi\rangle + |B_2\rangle Z|\psi\rangle + |B_3\rangle X|\psi\rangle + |B_4\rangle XZ|\psi\rangle}{2}$$

$|B_1\rangle|\psi\rangle$ with probability $\frac{1}{4}$;

$|B_2\rangle Z|\psi\rangle$ with probability $\frac{1}{4}$;

$|B_3\rangle X|\psi\rangle$ with probability $\frac{1}{4}$; and

$|B_4\rangle XZ|\psi\rangle$ with probability $\frac{1}{4}$.

Why teleportation doesn't allow FTL communication

Alice

Bob

Bob's final reduced density matrix is thus

$$\rho_B' = \frac{\mathrm{tr}_A\left(|B_1\rangle\langle B_1| \otimes |\psi\rangle\langle\psi| + \ldots\right)}{4}$$

$$= \frac{|\psi\rangle\langle\psi| + Z|\psi\rangle\langle\psi|Z + X|\psi\rangle\langle\psi|X + XZ|\psi\rangle\langle\psi|ZX}{4}$$

$$= \frac{\begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix} + \begin{bmatrix} |\alpha|^2 & -\alpha\beta^* \\ -\alpha^*\beta & |\beta|^2 \end{bmatrix} + \begin{bmatrix} |\beta|^2 & \alpha^*\beta \\ \alpha\beta^* & |\alpha|^2 \end{bmatrix} + \begin{bmatrix} |\beta|^2 & -\alpha^*\beta \\ -\alpha\beta^* & |\alpha|^2 \end{bmatrix}}{4}$$

$$= \frac{I}{2}$$

Why teleportation doesn't allow FTL communication

Alice  Bob

Bob's reduced density matrix after Alice's measurement is the same as it was before, so the statistics of any measurement Bob can do on his system will be the same after Alice's measurement as before!

Fidelity measures for quantum gates

Research problem: Find a measure quantifying how well a noisy quantum gate works that has the following properties:

It should have a simple, clear, unambiguous operational interpretation.

It should have a clear meaning in an experimental context, and be relatively easy to measure in a stable fashion.

It should have "nice" mathematical properties that facilitate understanding processes like quantum error-correction.

Candidates abound, but nobody has clearly obtained a synthesis of all these properties. It'd be good to do so!