

## 5 An introduction to Yang-Mills theory

### 5.1 Introduction

How can we construct successful new scientific theories? If there's some phenomenon in the Universe that current theories don't seem to explain — like dark matter, dark energy, neutrino oscillations or quantum gravity — it's tempting to throw out our current theories, and start over from scratch. Unfortunately, this gives us too much freedom in constructing new theories. Historically, a more successful approach has been to use existing theories to identify useful overarching principles that can guide the development of new theories. So, for example, classical mechanics led to the principle of conservation of energy and the ideas of Hamiltonian mechanics, ideas that played an important role in the development of quantum mechanics, even as classical mechanics itself was superseded.

Since 1954, one of the most important guiding principles in physics has been that our description of the world should be based on a special type of classical field theory known as a *Yang-Mills theory*. With the exception of gravitation, all the important theories of modern physics are quantized versions of Yang-Mills theories. These include quantum electrodynamics, the electroweak theory of Salam and Weinberg, the standard model of particle physics, and the GUTs (grand unified theories) proposed in the 1970s as extensions of the standard model.

The most important of these theories is the standard model of particle physics, which is our current best theory of how matter works. People sometimes describe the standard model as a Yang-Mills theory with an  $U(1) \times SU(2) \times SU(3)$  gauge symmetry. The purpose of these notes is to explain what this statement means. In particular, I will explain what a (classical) Yang-Mills theory is, and what it means to have a gauge symmetry. I won't explain the standard model itself, since it requires a detailed discussion of how to quantize a field theory, which would take us too far afield. However, the treatment here should leave you well prepared to understand the standard model and related ideas such as GUTs.

The notes assume a fair bit of background, and are aimed at graduate-level (or above) physicists or mathematicians with no prior exposure to the Yang-Mills equations. I assume you are comfortable with special relativity and Minkowski space, and with the relativistic formulation of Maxwell's equations, including concepts such as the Faraday tensor, and the current and potential four-vectors. You should be comfortable with elementary groups such as  $SU(n)$ , and with the idea of group representations, although we won't be using any sophisticated group theory or group representation theory. Finally, you'll need to be comfortable with calculus on curved surfaces. You won't need to know differential geometry, although the going will be easier if you have some prior exposure to differential geometry, such as is given in a course on general relativity.

**History:** The history of Yang-Mills theory is long and twisted. Many of the core ideas were developed independently by physicists and mathematicians, for completely different reasons, and it wasn't until the 1970s that the links between the two points of

view were worked out.

In physics, the first example of a Yang-Mills theory was Maxwell's theory of electromagnetism. However, Maxwell and his contemporaries had no idea what a Yang-Mills theory is, and certainly didn't think of the Maxwell equations in this way! It wasn't until much later that the idea of a gauge symmetry was formulated, and it became understood that Maxwell's equations satisfy such a symmetry. And it wasn't until later still that Yang-Mills theories were introduced as a large class of theories satisfying gauge symmetries.

Before the discovery of gauge symmetry and Yang-Mills theory, several people, including Lorentz, Einstein, and Poincare had studied the symmetries in Maxwell's equations. They discovered an unexpected symmetry, the Lorentz symmetry, which of course lies at the heart of special relativity. This led other people to investigate whether there are further symmetries of Maxwell's equations, and Weyl<sup>2</sup> discovered a new symmetry of electromagnetism, now known as gauge symmetry.

Along a different track of development, and a little earlier, Einstein discovered his general theory of relativity. One of the key ideas that helped Einstein write down the field equations of general relativity was a symmetry principle, namely, the idea that the field equations should be the same in every co-ordinate system. In the modern point of view, this too is an example of a gauge symmetry.

In a famous 1954 paper, Yang and Mills proposed a large class of classical field theories generalizing and inspired by electromagnetism, and satisfying a generalized type of gauge symmetry. When quantized, these Yang-Mills theories became the mainstay for developments in particle physics in the second half of the twentieth century. As noted above, examples of quantized Yang-Mills theories include many of our most important and successful physical theories, including quantum electrodynamics, the electroweak theory, the standard model of particle physics, and the GUTs (grand unified theories). Crucially, however, although general relativity satisfies a gauge symmetry, it is *not* known whether it is possible to cast general relativity as a Yang-Mills theory. This is highly unfortunate, since we understand how to quantize Yang-Mills theories, but not general gauge theories!

All the successful quantized Yang-Mills theories listed in the last paragraph follow the same general plan. We start from the assumption that the correct theory of the world is a quantized Lorentz-invariant Yang-Mills theory, and that all that has to be specified is the exact nature of the gauge symmetry. This is what the " $U(1) \times SU(2) \times SU(3)$ " is all about in the description of the gauge symmetry — it is a group which specifies the nature of the gauge symmetry. With the gauge symmetry fixed, the classical Yang-Mills theory is completely determined, and by quantizing it one can obtain the standard model<sup>3</sup>.

Once you've read the notes, you should understand the basic equations of Yang-Mills

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<sup>2</sup>I believe it was Weyl. I haven't read the original papers, and am relying on hearsay.

<sup>3</sup>Almost. There is one extra ingredient — the Higgs field — that I believe needs to be added in by hand. But this procedure gives you most of the structure of the standard model.

theory. In particular, I'll explain how you can start with a representation of a group,  $G$ , and construct the corresponding Yang-Mills theory. As an example, I'll explain how Maxwell's equations can be regarded as a Yang-Mills theory with gauge group  $U(1)$ . I won't explain the  $U(1) \times SU(2) \times SU(3)$  Yang-Mills theory in any detail, but in principle it is easy to construct using the recipe I will explain.

Now, there is a certain sense in which all this material is pretty easy to explain. I could just write the Yang-Mills equations out in full detail, and in some sense you'd "understand" Yang-Mills theory. However, merely knowing the equations is not the same as having a deep understanding of them. In the case of Yang-Mills theory, a deeper understanding is obtained if you understand certain geometric ideas behind Yang-Mills theory, and so I'm going to explain much of the geometric context. One reason for doing this is that the standard model is only one of two great theories of modern physics — the theory of gravitation (general relativity) is the other — and we don't know how to put the two together. However, both Yang-Mills theories and general relativity can be understood using very similar (though not quite the same) geometric ideas, and so it seems worth trying to understand them both from a geometric point of view.

Taking this geometric point of view means we need to work harder than if we just write the equations down directly. In particular, I'm going to explain quite a bit of background in differential geometry. The end result is, I hope, a deeper understanding of and appreciation for Yang-Mills theory than if we had simply started with the equations alone. Of course, this does not mean that we will obtain a comprehensive understanding of the Yang-Mills equations. Far from it — such an understanding cannot possibly be obtained by reading a short set of notes on the subject. This should not be surprising, since the Yang-Mills equations generalize Maxwell's equations, and understanding Maxwell's equations even passingly well requires years of work. To develop a better understanding, you must study them in much greater depth, and in particular understand them from multiple points of view, not just the geometric point of view we take here.

Having expounded on the benefits of geometry, let me note that if I were to rewrite these notes, I would probably not take the geometric viewpoint! While this is a very useful and powerful point of view, it has three significant disadvantages: (1) I don't think it is the simplest and most direct way of seeing Yang-Mills theory; (2) it burdens the beginner with a heavy load of definitions from differential geometry; and (3) the definitions can make it more difficult to see the physical forest for all the mathematical trees. From an expository point of view, one is forced to compromise between stating the definitions as quickly as possible, and a more lengthy exposition that fully motivates the definitions. Unfortunately, the definitions of modern differential geometry were arrived at over a painful process that took approximately 100 years, and understanding their motivation in depth is a lengthy task.

Despite all this, I still believe the geometric point of view is well worth studying, and that the notes may therefore be very useful! But in the future I hope to rewrite them from a rather different point of view, perhaps as part of a more extended treatment.

**What we don't cover:** Before jumping into a description of Yang-Mills theories, let me list a few important topics related to Yang-Mills theories that I'm not going to talk about.

First, although I write down the basic equations of Yang-Mills theory, I *won't* be deriving them from a Lagrangian (i.e., least-action) formulation. The ambitious reader familiar with least action principles might like to try deriving the Lagrangian formulation as an exercise. However, I decided that a full exposition of these ideas would add too much extra length to the article.

Second, everything we do is *classical*. To get to the standard model or the other quantum field theories of most interest in modern physics, we need to quantize the theory. This is, of course, a substantial topic in its own right, and is covered in any text on quantum field theory.

Third, the Yang-Mills theories that we construct are theories with a tremendous amount of symmetry. One of the most famous results in physics is *Noether's theorem*, which links continuous symmetries to conserved quantities in the system. We won't be following this link up, even though it cries out to be explored in detail.

Fourth, a critical point about Yang-Mills theories is that they are examples of *renormalizable* theories. Without explaining what renormalizability is all about, I will just say that it is a critical property for a field theory to have if it is to be useful for computing finite quantities<sup>4</sup>.

**Acknowledgements:** I learnt about gauge theory and Yang-Mills theory from the beautiful book “Gauge Fields, Knots and Gravity”, by John Baez and Javier P. Muniain [1].

## 5.2 Overview

The Yang-Mills equations are two easily-stated equations:

$$d_D F = 0 \tag{1}$$

$$*d_D * F = J. \tag{2}$$

Of course, while they may be easy to write down, that doesn't mean it's easy to understand what all those symbols means! In this section we develop a conceptual roadmap to understanding the Yang-Mills equations. We'll give a rough description of each element in the Yang-Mills equations, and hopefully convey something of their overall flavour. You shouldn't expect to completely understand everything in this section on a first read. Rather, the purpose is to begin forming a bird's-eye view that can assist in keeping oriented as we move through our detailed later discussions of each of the individual elements.

An unseen but important element of the Yang-Mills equations is the manifold on which all the action take place — this is where our physical objects live. In the case

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<sup>4</sup>I believe that it was for providing evidence that Yang-Mills theories are renormalizable that won Gerardus 't Hooft and Martin Veltman the 2003 Nobel Prize for Physics. However, I haven't read up on the details of this, and so can't be sure.

of both electromagnetism and the standard model, that manifold is just the Minkowski space of special relativity. We'll explain the equations on a general manifold, since it entails little additional complexity, but if you prefer you can always imagine that we are working on Minkowski space.

**The connection,  $D$ :** The fundamental physical object in the Yang-Mills equations is the *connection*, which is denoted  $D$ . This shows up only surreptitiously in the equations, as a subscript, albeit twice, but the connection actually determines nearly all the other quantities in the equations, including  $d_D$ ,  $F$ , and all the starred quantities, and only excluding the current,  $J$ , which describes the distribution and velocity of charges in the theory. Thus, if the current is regarded as given, then the Yang-Mills equations are really just a set of equations that constrain the connection,  $D$ .

Mathematically, the connection tells us how to move stuff around from point to point on the manifold — it's a means for achieving so-called “parallel transport” on the manifold. If you've taken a course in general relativity you've met an example of a connection; the notion we'll be talking about is essentially the same idea, although we need it in a more general form than how it's used in general relativity.

Physically, the connection is the fundamental physical field of the theory. Together with the current,  $J$ , the connection completely determines all the physical properties of the system. In the case where the Yang-Mills theory is Maxwell's equations, the connection is essentially just the electromagnetic vector potential.

It should be apparent to you from this description that the mathematical and physical meanings of the connection are very different! So far as I am aware, there is no good reason known why we should expect these two notions to coincide. Stated another way, the physicists don't know why their notion of a field should also play a role in the description of parallel transport, and the mathematicians don't know why the object they use to describe parallel transport should have any physical significance as a field. Indeed, the co-inventor of Yang-Mills theories, Yang, has recalled that he didn't even learn what a connection is until 1975, twenty-one years after he and Mills proposed Yang-Mills theory. From his point of view, he was simply writing down a physical field to generalize the vector potential of electromagnetism.

**The curvature,  $F$ :** We've seen that the connection plays a role analogous to the vector potential in Maxwell's equations. Of course, we don't ordinarily write Maxwell's equations directly in terms of the vector potential, but rather work in terms of derived quantities, namely the electric and magnetic fields, or, if we are a bit more sophisticated, in terms of the Faraday tensor.

In Yang-Mills theory, the Faraday tensor is generalized to the *curvature*,  $F$ . Mathematically, the curvature is derived from the connection essentially by taking commutators of certain differential operators related to the connection. Physically, as already stated, you should think of the curvature as generalizing the Faraday tensor in electromagnetism; it (rather than  $D$ ) is what most directly acts on physical particles.

If you've met the concept of curvature previously, then you probably know that the way we ordinarily conceive of Minkowski space it is a flat (i.e., uncurved) space.

However, the idea of Yang-Mills theory is to throw out the ordinary concept of flat Minkowski space. What the connection (e.g., the vector potential) does is to introduce a “twist” into Minkowski space. The curvature  $F$  then measures the extent to which this twist causes a deviation from the ordinary flat geometry of Minkowski space.

**The exterior covariant derivative,  $d_D$ :** The operation  $d_D$  is known as the *exterior covariant derivative*. Very roughly speaking, the quantity  $d_DX$  is a measure of how fast  $X$  is changing as it is moved around in various directions on the manifold. The Yang-Mills equations thus tell us that, in a certain sense,  $F$  is not changing as we move around the manifold, while the way  $*F$  changes is determined by the current,  $J$ .

Now, this is pretty clearly nonsense! After all, in electromagnetism the Faraday tensor  $F$  certainly can change. However, recall above that I said that the connection causes a “twist” in Minkowski space. In fact, in some sense the Yang-Mills equation  $d_DF = 0$  is telling us that, with respect to this twist,  $F$  is not changing<sup>5</sup>.

**The Hodge \*-operation:** The final element in the Yang-Mills equations is the *Hodge \*-operation*. This is applied twice in the second Yang-Mills equation, first to  $F$  to obtain  $*F$ , and then to  $d_D *F$  to obtain  $*d_D *F$ . To understand the role of the Hodge-\*, recall that in the conventional Maxwell equations it is possible to interchange the  $E$  and  $B$  fields, provided one ignores the source  $\rho$  and current  $j$  terms, and makes suitable sign changes and rescaling. The Hodge \*-operation is a generalization, interchanging certain temporal degrees of freedom (generalizing the  $E$  field) and certain spatial degrees of freedom (generalizing the  $B$  field).

**What about the Lorentz force law?** As we have seen, the Yang-Mills equations generalize the Maxwell equations for electromagnetism. However, the Maxwell equations are not, on their own, a complete set of equations for a physical theory. They must be supplemented by an additional law, the Lorentz force law, that tells us how charged particles accelerate in an electromagnetic field. The Lorentz force law for a particle of mass  $m$  and charge  $q$  is

$$m\dot{u}^j = qF^j_k u^k, \tag{3}$$

where  $u^j$  are the components of the velocity four-vector,  $\dot{u}^j$  indicates the derivative with respect to proper time, and  $F^j_k$  are the components of the Faraday tensor. This equation cannot be deduced from the Maxwell equations. This can be seen since the mass plays a critical role in the Lorentz force law, while it is completely absent from the Maxwell equations.

While the Maxwell equations don’t specify a complete physical theory, the Maxwell-Lorentz equations do. In particular, if we are given an initial configuration of charges, currents and fields, then the Maxwell-Lorentz equations tell us the time rate of change of all quantities, and so can in principle be integrated forward in time to tell us the

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<sup>5</sup>Actually, it’s a bit more complicated than that. Consider the equation  $\nabla \cdot B = 0$  from Maxwell’s equations. This doesn’t tell us that  $B$  is not changing, just that the total flux of  $B$  from a volume is always zero. The equation  $d_DF = 0$  has more of this nature.

configuration at later times<sup>6</sup>.

Can we generalize the Lorentz force law so that it applies also to Yang-Mills theories? Unfortunately, although such a generalization is certainly known, it is not (yet) known by me! The obvious generalization is to simply replace the Faraday tensor by the curvature. This certainly does not work, as we shall see later that the curvature acts on the wrong space. Finding a suitable generalization could certainly be done by starting with one of the standard Lagrangians describing the interaction of a Yang-Mills field with matter (e.g., such as arise in the standard model), but I haven't actually gone through and worked out the details.

Incidentally, I said above that particles most directly feel curvature, rather than the connection. It should be clear from the discussion in the last paragraph that I was blowing hot air with that statement (excusably, I hope). Without a generalization of the Lorentz force law to arbitrary Yang-Mills theories, we can't say this for sure. Nonetheless, I hope you will agree that it seems highly likely.

**A warning about Yang-Mills theories:** The Maxwell-Lorentz equations have some significant problems as theories of physics, and it seems reasonable to suppose that similar problems occur in most other Yang-Mills theories. Particularly problematic in electromagnetism is what happens when you look up close at a charge. Consider, for example, an electron. How should we model the electron in the Maxwell-Lorentz theory of electromagnetism?

One way is to imagine that the electron is a simple point charge. Unfortunately, the Coulomb law tells us that there must be a singularity in the field at the point the electron sits at. The field isn't defined at that point, and the Lorentz force law cannot be applied; the theory doesn't tell us which way the electron should move! Although various *ad hoc* fixes to this problem can be applied, and in practice are, there does not seem to be a conceptually clean and simple way of modifying the theory so this problem is avoided.

A second possibility is to imagine that the electron is a smeared out ball of charge. This solves the problem with the singularity in the field, but raises the problem of why the electron doesn't fly apart due to the tremendous internal repulsive forces? One might try to solve this problem by positing the existing of countervailing forces that hold the electron together, but I do not know of a successful and simple theory in this vein.

All this should make you uneasy. After all, the standard model — one of the greatest achievements of science — is a quantized Yang-Mills theory. We've obtained the standard model by starting with a theory with serious conceptual problems, then applying an *ad hoc* quantization procedure to that theory. Lo, by some process of alchemy a beautiful theory is revealed, explaining most of what we see in the world. Now, this procedure of quantizing Yang-Mills theories has been so fantastically successful that it

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<sup>6</sup>I am, as physicists often do, ignoring all questions of whether or not singularities might arise that make it impossible to continue the process of integration, merely assuming that these equations are sufficiently nice that this will never be the case.

deserves to be taken very seriously as a means of finding new theories. But that certainly doesn't mean that you should be comfortable with it!

**Exercise 5.1:** One possible idea for understanding the structure of the electron is to suppose that it is gravitation which is holding the electron together against the internal forces of repulsion. Show that this doesn't work, that gravitation is far too weak to be the force responsible for holding the electron together internally.

**Why have Yang-Mills theories been so successful?** To answer this question, it is necessary to think about the more fundamental question of how one should study dynamics. Naively, it is tempting to study dynamics directly, trying to directly observe the effect of the forces that hold stuff together. This approach has, of course, been very successful, and has a long tradition going back to phenomena such as Newton's discovery of the inverse square law (which has its roots in the work of Brahe, Kepler, and Galileo), and is still used even today.

The twentieth century brought another idea for studying forces to the fore. This is the idea that one doesn't need to study them directly, but instead can study symmetries of the theory, and that if there is enough symmetry those symmetries effectively dictate the interaction.

In simple forms, this is, of course, an old idea — we know that if two particles are the “same” then quantities such as the Hamiltonian or Lagrangian should be invariant under interchange of the two particles, and this constrains the form of those quantities.

However, this idea can be taken much further. In particular, if we choose a sufficiently large class of symmetries, then this greatly constrains the class of physical theories that are allowed. In some cases, this constraint is so great that symmetry plus a few other simple physical principles becomes sufficient to uniquely pick out our theory. This was, for example, the case for Einstein's general theory of relativity.

How do the Yang-Mills equations fit into this picture? Essentially, they can be viewed as a machine which takes as input a heavy symmetry constraint — the constraint of invariance with respect to some gauge symmetry that we specify — and produces as output a classical field theory satisfying that constraint. Furthermore, that classical field theory turns out to have many other desirable features, such as renormalizability. Thus, the Yang-Mills approach offers an excellent sandbox for constructing interesting theories. It is broad and rich, in that we can specify many different types of symmetry group, and thus describe many different types of physics. At the same time, Yang-Mills theories have properties such as renormalizability which make them suitable for use in a quantized description.

### 5.3 Background on differential geometry

We are taking a geometric approach to the Yang-Mills equations. This poses an expository challenge, for I do *not* wish to assume you are familiar with differential geometry. Unfortunately, understanding even elementary differential geometry requires mastering



a large body of definitions. In some sense, the most efficient way of mastering this material is simply to proceed through 40 or 50 pages of definitions, covering topological spaces, smooth manifolds, tangent vectors and so on, with occasional lemmas and theorems mostly of technical interest. Most books on differential geometry proceed in this fashion before getting to the truly interesting theorems of differential geometry.

In my opinion this approach is not suitable for us for two reasons.

First, on general grounds, I don't believe this long line of definitions is a good approach. Definitions are not made in a vacuum, they have a historical context and reason, often being made in response to some mathematical or scientific problem, or because someone sees a clever way of generalizing earlier results. They are almost always inextricably linked to the reasoning used in theorems; often alternate definitions are tried and dropped when it is realized that they don't have quite the right properties to make the line of reasoning in the proof of some theorem or another work. In the case of differential geometry, it took approximately 100 years for the definitions of "elementary" differential geometry to take their modern form, spanning from the mid 19'th century to the mid 20'th century. In the standard expository approach to differential geometry we see the endpoint of this, but only barely glimpse the underlying motivations, all the alternate definitions, now discarded, or the clever way in which some definition is tailored to a theorem proved one hundred pages later.

Second, our object here is not to become experts in differential geometry. Rather, it is to understand the basic ideas of Yang-Mills theory, and the differential geometry is simply a means to that end. Going through all the detailed definitions of differential geometry leaves us in danger of losing sight of our principal purpose.

In view of this, I'm going to cheat a little. Rather than give fully rigorous definitions for concepts like manifolds, tangent spaces and so on, in this section I introduce the required background informally, stating the main ideas, and giving some examples, and feeling free to ignore issues like smoothness and so on<sup>7</sup>. This background should be sufficient to build up a detailed understanding of objects such as the connection and curvature, which are central to Yang-Mills theory.

Despite this approach, we still have a sizeable body of definitions to cover, and there are a few points where I'll need to gloss over some details — I'll warn you when I do this. Nonetheless, I hope this informal approach will let you more quickly appreciate the Yang-Mills equations, and give you a good feeling for the geometric context in which they arise.

**Manifolds:** The arena for Yang-Mills theory is a general (smooth) manifold, which we shall denote generically by  $M$ . Manifolds are generalizations of the familiar space  $\mathbb{R}^n$ . They can be thought of as smooth surfaces that look locally like  $\mathbb{R}^n$ , but which may have quite a different global structure. For example, the surface of the ordinary sphere looks locally like  $\mathbb{R}^2$  — this is why people used to think the world is flat — but

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<sup>7</sup>Contrary to what physicists are sometimes taught as undergraduates, these issues are important. But they are subsidiary to our main goal here, which is to understand the broad sweep of Yang-Mills theory, and so we'll ignore them.

of course is really quite different from  $\mathbb{R}^2$ .

Now, all our examples of Yang-Mills theory — and all the applications of Yang-Mills of most importance in modern physics — are for the case when the manifold  $M$  is Minkowski space, which should already be familiar to you from special relativity. Thus, if you simply replace the term “manifold” in all that follows by “Minkowski space”, you should be able to follow what is going on.

However, to understand the motivation for some of the definitions we make it is useful to have a second example manifold in mind. An excellent choice is to choose  $M$  to be the surface  $S^2$  of a sphere in three dimensions. (Be warned: although I said “in three dimensions”, of course the surface of the sphere is a two-dimensional manifold, because it looks locally like  $\mathbb{R}^2$ . This is why it is denoted  $S^2$ .) If you keep both the example of Minkowski space and  $S^2$  in mind as we discuss manifolds, you should be well placed to understand all the further definitions that we make.

**Local co-ordinate systems:** Perhaps the most important concept for us in discussing manifolds is the idea of local co-ordinates. Local co-ordinates are, as their name indicates, a means of describing location in some small region of a manifold. For example, on a two-dimensional manifold such as  $S^2$ , around any point  $x$  on the sphere we can find a small surrounding neighbourhood  $N$  which can be given local co-ordinates — a map between points in  $N$  and co-ordinates  $(x^1, x^2)$  in  $\mathbb{R}^2$ . These co-ordinates uniquely identify points in the neighbourhood  $N$ . Such local co-ordinate systems are a great convenience for making definitions, and for doing calculations, and we shall often find it convenient to introduce a local co-ordinate system.

Incidentally, local co-ordinate representations sometimes get a bad rap in the literature. Some authors go to extremes in attempting to perform calculations without the use of local co-ordinates, sometimes adopting an evangelical tone in advocating this approach. This, in my opinion, makes as much sense as insisting that you navigate around a city without using street numbers: “make a right after the corner store, then a left at the burnt-out tree stump, three blocks up, right again, ...”. Co-ordinates are simply another tool — use them if you think they’ll be helpful.

**Scalar fields:** We’re going to consider many different types of *fields* on the manifold. As you can no doubt guess, a field simply associates to each point on the manifold another object — maybe a number, maybe a vector, or perhaps some other types of quantity — often intended to represent some physical quantity. So, for example, a temperature field defined on the manifold  $\mathbb{R}^3$  might be used to model temperature in the atmosphere. The simplest type of field is a *scalar field*, which assigns to each point  $x$  on the manifold a number  $s(x)$ . We’ll find it useful to consider both real and complex scalar fields.

**Tangent vectors:** One of the most common types of field is a tangent vector field, which assigns to each point  $x$  on a manifold a vector  $v(x)$  which is tangent to the manifold. Such tangent vectors can be used to represent a direction of travel on the manifold; a natural physical example of a tangent vector field is the velocity field inside a fluid.

Unfortunately, rigorously defining and studying the elementary properties of tangent vectors on a general manifold requires a fair bit of tedious work.

Fortunately, our intuition about tangent vectors is already excellent; the rigorous approach mostly reveals facts already well known to your intuition. For our purposes, it will therefore suffice to think of the example of  $S^2$ . In particular, we all know what it means for a vector to be tangent to  $S^2$  at some point  $x$  on  $S^2$ .

Tangent vectors are easily given co-ordinate representations. If  $(x^1, x^2)$  are local co-ordinates on the manifold defined in a neighbourhood of  $x$ , then we see that there are natural associated tangent vectors  $e_1$  and  $e_2$ , representing motion, respectively, in the direction of increasing  $x^1$  at unit speed, and increasing  $x^2$ , again at unit speed. A general tangent vector  $v$  at the point  $x$  can be written as a linear combination  $v = v^1 e_1 + v^2 e_2$ , with the direction and speed of motion obvious. We refer to  $v^1$  and  $v^2$  as the co-ordinates for  $v$  with respect to the local co-ordinates  $(x^1, x^2)$ . It is often convenient to abbreviate the local co-ordinates to just  $x^j$ , and the corresponding co-ordinates for the tangent vector to  $v^j$ .

The space of all vectors tangent to a given point  $x \in M$  is obviously a vector space, and we denote it by  $T_x M$ . If  $M$  is an  $n$ -dimensional manifold, then the tangent space  $T_x M$  also has  $n$  dimensions for all points  $x$ . So, for example, the tangent spaces  $T_x S^2$  to the sphere are all two-dimensional; we can think of  $T_x S^2$  as the plane tangent to  $S^2$  at the point  $x$ .

With these concepts in mind, it is now easy to define a *vector field* on a manifold  $M$  as a function assigning to each point  $x$  on the manifold a vector  $v(x)$  in the tangent space  $T_x M$ .

As we have defined them, vector and scalar fields are defined everywhere on the manifold. In fact, quite often when dealing with local co-ordinates we will find it convenient to consider a field (vector, scalar, or any other type of field) as being defined just on the neighbourhood in which the co-ordinates are defined.

**The commutator of vector fields:** Suppose  $v$  is a tangent vector at a point  $x$  on a manifold  $M$ . Suppose  $v^j$  are the components of  $v$  with respect to some local co-ordinate system  $x^j$ . It is possible to associate with  $v$  a differential operator via the correspondence

$$v \Leftrightarrow v^j \frac{\partial}{\partial x^j}, \tag{4}$$

where we use the Einstein summation convention. The differential operator on the right acts on functions defined in the neighbourhood of  $x$  in the obvious way, so we have

$$v(f) = v^j \frac{\partial f}{\partial x^j}. \tag{5}$$

Why make this correspondence between vectors and differential operators? Unfortunately, answering that question fully would take me too far afield, and so I will ask you merely to accept that the correspondence is useful. The main consequence for us

is that it allows us to define a sensible notion of a commutator between vector fields  $v$  and  $w$ . In particular, we can define

$$[v, w](f) \equiv v(w(f)) - w(v(f)). \quad (6)$$

It is clear that this defines a differential operator  $[v, w]$  acting on functions. *A priori* one might think that this differential operator contained both first- and second-order derivatives, but in fact some elementary algebra shows that the second-order terms all cancel, leaving just first-order derivatives. Thus, using the correspondence of Eq. (4) in the other direction we may regard  $[v, w]$  as also defining a vector field. A calculation shows that this vector field has components:

$$[v, w]^k = v^j \frac{\partial w^k}{\partial x^j} - w^j \frac{\partial v^k}{\partial x^j}. \quad (7)$$

**Exercise 5.2:** Verify the co-ordinate representation of Eq. (7).

**The co-ordinate vector fields:** Suppose we have co-ordinates  $x^j$  defined on some neighbourhood on the manifold. As we noted above, at any point  $x$  in the neighbourhood, there is a tangent vector  $e_j \in T_x M$  pointing in the  $x^j$  direction on the manifold, and denoting travel at unit speed. It will be convenient to introduce a new notation,  $\partial_j \equiv e_j$ , for this tangent vector, motivated by the correspondence with the differential operator,  $\partial/\partial x^j$ ; indeed, we shall also use  $\partial_j$  as shorthand for  $\partial/\partial x^j$ . Note that we can expand any vector field at any point  $x$  in the neighbourhood as  $v = v^j \partial_j$ , where the  $v^j$  are coefficients that depend on  $x$ , and we make use of the Einstein summation convention. As a result, we call these vector fields a *co-ordinate basis* or *co-ordinate vector fields* for the neighbourhood. For later use it is useful to note that the co-ordinate basis commutes:

$$[\partial_j, \partial_k] = 0. \quad (8)$$

This follows immediately from the representation of Eq. (7).

**Exercise 5.3:** Suppose  $x^j$  and  $\tilde{x}^j$  are two different co-ordinate systems, both defined in neighbourhoods of a point  $x$ . Let  $v$  be a vector in the tangent space  $T_x M$ . Convince yourself that the components  $v^j$  and  $\tilde{v}^j$  corresponding to the co-ordinate bases  $\partial_j$  and  $\tilde{\partial}_j$  are related by

$$\tilde{v}^k = \frac{\partial \tilde{x}^k}{\partial x^j} v^j. \quad (9)$$

A useful shorthand notation for this transformation law is

$$\tilde{v}^k = (\partial_j \tilde{x}^k) v^j. \quad (10)$$

The inverse transform is clearly

$$v^k = \frac{\partial x^k}{\partial \tilde{x}^j} v^j = (\tilde{\partial}_j x^k) v^j, \quad (11)$$

where  $\tilde{\partial}_j \equiv \partial/\partial \tilde{x}^j$ . Note that since we haven't rigorously defined tangent vectors, it's not possible to provide a rigorous proof of these equations. Nonetheless, you should be able to convince yourself pretty thoroughly that they are true. Two approaches you might take are: (1) to look at examples of different co-ordinate systems near a point on the sphere; and (2) to start from the point of view in which tangent vectors are identified with differential operators.

**Cotangent space:** To each tangent space  $T_x M$  we associate a *cotangent space*, which we denote  $T_x^* M$ . This is the vector space dual to  $T_x M$ , and consists of linear maps from  $T_x M$  into  $\mathbb{R}$ . Just as  $T_x M$  has as a basis the vectors  $\partial_j$ , the cotangent space has as a basis dual vectors  $dx^j$  whose action is defined by

$$dx^j(\partial_k) \equiv \delta_k^j. \quad (12)$$

Upon first sight, the notation  $dx^j$  may seem mysterious. It is tempting to think that it must be associated to calculus in some way, and perhaps represents an infinitesimal. In fact, from the right point of view, there *is* a close connection to the calculus notions. We'll begin developing this point of view in Section 5.5, but we won't develop it to its fullest conclusion. For our purposes it's best to think of  $dx^j$  as simply being a vector in the space  $T_x^* M$ , and to regard the use of the infinitesimal notation as a coincidence to be ignored.

**One-forms:** We saw earlier that a vector field is a function that associates to each point  $x$  on the manifold a point  $v(x)$  in the tangent space  $T_x M$ . The analogous notion for the cotangent space is that of a *one-form*. By definition, a one-form is a map  $\omega$  which takes points  $x$  on the manifold  $M$  to an element  $\omega(x)$  of the corresponding cotangent space,  $T_x^* M$ .

So far, we've defined three types of field — scalar fields, vector fields, and one-forms. This may seem a lot, but Yang-Mills theory requires two further generalizations of the field concept, known respectively as  $p$ -forms (unsurprisingly, they generalize one-forms), and vector bundles. We'll explain  $p$ -forms now, and vector bundles later, in Section 5.7. In Section 5.10 we'll define a notion of a vector bundle-valued  $p$ -form that generalizes and combines both  $p$ -forms and vector bundles; it is these vector bundle-valued  $p$ -forms that are the natural objects in Yang-Mills theory.

**The wedge product:** To explain  $p$ -forms, we need to turn away from the context of differential geometry, to vector spaces. Suppose  $v_1, v_2 \in V$  are two elements of a vector space  $V$ . We define the *wedge product* of  $v_1$  and  $v_2$  by

$$v_1 \wedge v_2 \equiv \frac{v_1 \otimes v_2 - v_2 \otimes v_1}{2}. \quad (13)$$

We can extend this definition to more vectors, e.g.,  $v_1, \dots, v_p \in V$ , by

$$v_1 \wedge v_2 \wedge \dots \wedge v_p \equiv \frac{\sum_{\pi} \text{sgn}(\pi) v_{\pi(1)} \otimes v_{\pi(2)} \otimes \dots \otimes v_{\pi(p)}}{p!}, \quad (14)$$

where the sum is over all permutations  $\pi$  of  $1, \dots, p$ , and  $\text{sgn}(\pi)$  is  $+1$  if  $\pi$  is an even permutation, and  $-1$  if  $\pi$  is an odd permutation. Notice that with this definition, we have  $v_1 \wedge v_2 = -v_2 \wedge v_1$ . If we have more vectors, then swapping any two in the wedge product similarly produces a minus sign.

The  $p$ 'th exterior product of  $V$  is the vector space  $\Lambda^p(V)$  spanned by vectors of the form  $v_1 \wedge \dots \wedge v_p$ . By convention, we choose  $\Lambda^0(V) = \mathbb{R}$ , a one-dimensional vector space.

**Exercise 5.4:** Show that if any two vectors  $v_j$  and  $v_k$  in a wedge product  $v_1 \wedge \dots \wedge v_p$  are the same, then the wedge product vanishes.

**Exercise 5.5:** Show that  $\Lambda^p(V) = 0$  if  $p \geq \dim(V)$ .

**Exercise 5.6:** When  $0 \leq p \leq \dim(V)$ , show that the dimension of  $\Lambda^p(V)$  is  $\binom{\dim(V)}{p}$ .

**$p$ -forms:** We are now in position to define  $p$ -forms, generalizing our earlier definition of one-forms. To do so, we have to return to differential geometry, and the context of a manifold,  $M$ . A  $p$ -form is just a map  $\omega$  which takes each point  $x$  on the manifold to an element of  $\Lambda^p(T_x^*M)$ . In the case when  $p = 1$  this reduces to the definition of a one-form given above, since  $\Lambda^1(T_x^*M) = T_x^*M$ . A 0-form is just a scalar field, since  $\Lambda^0(T_x^*M) = \mathbb{R}$ , by convention.

Recall that we had a basis  $dx^j$  of cotangent vectors for the cotangent space  $T_x^*M$ . Thus, a one-form will look locally something like:

$$\omega = \omega_j dx^j, \quad (15)$$

where we call the  $\omega_j$  a local co-ordinate representation for the one-form. Similarly, a two-form can always be expanded in local co-ordinates like

$$\omega = \omega_{jk} dx^j \wedge dx^k. \quad (16)$$

We denote the set of all  $p$ -forms on a manifold  $M$  by  $\Omega^p(M)$ .

**Exercise 5.7:** Show that the coefficients  $\omega_{jk}$  in Eq. (16) can always be chosen so as to be antisymmetric, i.e.,  $\omega_{jk} = -\omega_{kj}$ . Furthermore, prove that this choice is unique.

## 5.4 The difficulty of doing calculus on manifolds

In applications of differential geometry to physics (like Yang-Mills theory) a typical setup is that the underlying manifold represents all of spacetime, and that fields on that manifold are used to describe physical quantities. The equations of physics then describe how the time variation of those physical quantities is related to their spatial variation.

To describe all of this in a precise mathematical fashion, obviously we need a way of doing calculus on manifolds. Unfortunately, it turns out to be not obvious how to do calculus on manifolds. To see why this is, let's get some idea of the problems that arise in trying to define the derivative of a vector field. In particular, suppose  $v(x)$  is a vector field, and we want to compute its derivative — its rate of change — as we move in some direction  $y$  away from the point  $x$  (so  $y \in T_x M$ ) on the manifold. The natural guess to make is that the derivative should be the  $\Delta \rightarrow 0$  limit of the expression

$$\frac{v(x + \Delta y) - v(x)}{\Delta}. \quad (17)$$

The first difficulty in understanding this expression is to understand what  $x + \Delta y$  means. Fortunately, with a little work it turns out that this difficulty can be resolved by working in a local co-ordinate representation for  $x$  and  $y$ . Much more serious is the fact that the vector  $v(x + \Delta y)$  lives in the tangent space  $T_{x+\Delta y}$ , while the vector  $v(x)$  lives in a *completely different* tangent space  $T_x M$ , and we have no *a priori* way of making sense of the difference of these two vectors.

Now, you might object, both vector spaces have the same dimension, surely we should be able to identify them? Well, okay, but *how* exactly should we do that? Vector spaces don't just come with labels automatically attached that let us identify them with one another. In fact, with a little bit of work we can specify a labelling scheme that lets us identify one tangent space with another, but it turns out to require some care and subtlety. This is the idea behind the definition of the connection or covariant derivative, which we'll study in Section 5.6.

Another way of seeing the difficulty is to examine the example of the surface of the sphere,  $S^2$ . Different points on the sphere have different tangent spaces, and it is clear that taking the difference between tangent vectors in different tangent spaces may end up resulting in a vector that belongs to neither tangent space. Now, you might object that at least we get a vector this way. But it should bother you that we've taken the derivative of a two-dimensional vector field, and ended up with a three-dimensional vector! What's worse, it turns out that the surface of the sphere,  $S^2$ , can actually be embedded in higher dimensional spaces in multiple ways, and so we don't get an unambiguous definition even in this way.

**Problem for the author 5.1:** While I've poured scorn on this method of taking derivatives by embedding in a higher-dimensional space, it may be worth investigating in more detail. In particular, it is known that any  $n$ -dimensional manifold

can be embedded in  $\mathbb{R}^{2n}$ . Might it be possible to construct some canonical embedding which allows us to define the derivative in an unambiguous way? What properties might this derivative have?

Because of these difficulties, defining a sensible notion of derivative is a challenge for elementary differential geometry. It turns out that there are two approaches to generalizing the derivative that have been found to be especially useful — the exterior derivative, which is defined in the next section, and the covariant derivative, also known as the connection, which is defined in Section 5.6. In Section 5.10 we'll define the exterior covariant derivative, which combines both these operations.

Why are two notions of derivative needed? I must admit, I don't understand the answer to this question as well as I would like. However, my imperfect understanding is that the definition of the covariant derivative is tailored to proving generalizations of the fundamental theorem of calculus and its relatives (such as Stokes', Gauss' and Green's theorems). We won't prove this generalization, but it has the form  $\int_M d\omega = \int_{\partial M} \omega$ , where  $\omega$  is a  $p$ -form,  $\partial M$  denotes the boundary of  $M$ , and  $d\omega$  is the exterior derivative of  $\omega$ . The covariant derivative is tailored to a different purpose. At this point I must admit, to some embarrassment, that it is not so clear to me what that purpose is<sup>8</sup>!

## 5.5 Exterior derivatives

Having discussed some of the complications of doing calculus on manifolds, let us now turn to our first approach to defining a notion of derivative on a manifold, namely the exterior derivative.

Suppose  $\omega$  is a  $p$ -form on a manifold  $M$ . We will define the *exterior derivative*  $d\omega$  of  $\omega$ , which is a  $p + 1$ -form on the manifold. To understand the definition and its motivation, it is best to start with a 0-form,  $f$ , i.e., a scalar field on the manifold. We define the exterior derivative of such a scalar field by

$$df \equiv \frac{\partial f}{\partial x^j} dx^j = (\partial_j f) dx^j, \quad (18)$$

where  $x^j$  is any local co-ordinate system, and it may help to recall that we are using the Einstein summation convention, and the notation  $\partial_j \equiv \partial/\partial x^j$ . It is not difficult to check that this definition does not depend on the choice of local co-ordinate system.

Intuitively,  $df$  tells us how fast  $f$  is changing as we move in different directions on the manifold. In particular, if we move in a direction specified by the tangent vector  $v$ , then  $df(v) = v^j \partial f / \partial x^j$  is just the rate of change of  $f$  as we move in that direction. As a result, this definition of the exterior derivative dovetails well with our earlier point of view in which a tangent vector  $v$  was regarded as a differential operator  $v^j \partial / \partial x^j$ . This gives  $v$  a way of acting on functions, via  $v(f) \equiv v^j \partial f / \partial x^j$ , and so we have  $v(f) = df(v)$ .

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<sup>8</sup>In fact, for the covariant derivative of a vector field on a Riemannian manifold, there is a well-known motivation in terms of geodesic motion. But for more general types of field, I don't know of a similar motivation.



**Exercise 5.8:** Show that the definition of Eq. (18) does not depend on the co-ordinate system being used.

**Exercise 5.9:** Suppose  $x^j$  are local co-ordinates defined on some neighbourhood on the manifold. Fix a particular co-ordinate,  $x^j$ , and regard it as a scalar field defined on the neighbourhood. Show that the exterior derivative  $dx^j$  of  $x^j$  is identically equal to the one-form  $dx^j$  defined by Eq. (12).

We now extend the definition of the covariant derivative so that it maps a  $p$ -form  $\omega \in \Omega^p(M)$  to a  $p+1$ -form  $d\omega \in \Omega^{p+1}(M)$ . The following rules are sufficient to do this:

1. When  $f$  is a 0-form, i.e., a scalar field,  $df$  is defined as in Eq. (18).
2.  $d$  is linear, i.e.,  $d(c^j \omega_j) = c^j d\omega_j$ , where  $c^j$  is any set of constants.
3.  $d(f\omega) = df \wedge \omega + f d\omega$ , where  $f$  is any 0-form.
4.  $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ , where  $\alpha$  is a  $p$ -form, and  $\beta$  is a  $q$ -form. Note that this rule can be regarded in a natural way as a generalization of the last rule.
5.  $d(d\omega) = 0$  for all  $\omega$ .

These rules are sufficient to compute the exterior derivative of any  $p$ -form. A general proof of this fact is less instructive than working through a good example, so let's work through such an example. In particular, suppose  $\omega$  is a two-form. Then in a local co-ordinate system it can be expanded as

$$\omega = \omega_{jk} dx^j \wedge dx^k. \quad (19)$$

Then by Rule 3 we have

$$d\omega = d\omega_{jk} \wedge dx^j \wedge dx^k + \omega_{jk} d(dx^j \wedge dx^k). \quad (20)$$

But  $d\omega_{jk} = (\partial_l \omega_{jk}) dx^l$  by Rule 1, and  $d(dx^j \wedge dx^k) = d(dx^j) \wedge dx^k - dx^j \wedge d(dx^k) = 0$ , by Rules 4 and 5. So we have:

$$d\omega = (\partial_l \omega_{jk}) dx^l \wedge dx^j \wedge dx^k. \quad (21)$$

In fact, instead of taking the axiomatic point of view described above, we could equally well have started with this as our *definition* for the exterior derivative, and then checked that it is well-defined independent of the co-ordinate system used (we'll do this in the exercises below), and satisfies the rules specified above.

The formula of Eq. (21) shows that there is an intuitive sense in which we can think of the exterior derivative  $d\omega$  as measuring the rate of change of  $\omega$ . In particular, if we let  $d\omega$  act in a natural way on a tangent vector  $v$  in the "first slot" (i.e., as the first argument), then we get the following  $p$ -form back,

$$d\omega(v) = (\partial_l \omega_{jk}) dx^l(v) \wedge dx^j \wedge dx^k \quad (22)$$

$$= v^l (\partial_l \omega_{jk}) dx^j \wedge dx^k. \quad (23)$$

This can be thought of as the rate of change of  $\omega$  as we move across the manifold in the  $v$  direction.

**Exercise 5.10:** Suppose  $\omega = \omega_j dx^j = \tilde{\omega}_j d\tilde{x}^j$  is a one-form that can be expanded in two different ways with respect to two different co-ordinate systems,  $x^j$  and  $\tilde{x}^j$ . This gives rise to two possible expressions for  $d\omega$ , either

$$d\omega = (\partial_k \omega_j) dx^k \wedge dx^j \quad (24)$$

or

$$d\omega = (\tilde{\partial}_k \tilde{\omega}_j) d\tilde{x}^k \wedge d\tilde{x}^j. \quad (25)$$

Show that these two expressions are identically equal.

**Exercise 5.11:** Show that rules 1-5 defined above uniquely specify the exterior derivative in a well-defined way.

## 5.6 Connections and the covariant derivative

The connection (also known as the covariant derivative) gives a second approach to defining a notion of derivative on a manifold. In particular, the connection gives us an explicit way of identifying a tangent vector  $v$  in one tangent space  $T_x M$  to a tangent vector  $v'$  in a different tangent space  $T_y M$ . The way this is done is to imagine “dragging” the tangent vector  $v$  along a path from  $x$  to  $y$ , all the time keeping it the “same”. Thus, if we can define a local notion of what it means for vectors in neighbouring tangent spaces to be the “same”, then we can extend it to a global notion.

The way we’ll define the local notion is to require that in *some* local co-ordinate system  $x^j$ , the components of the tangent vector  $v^j$  aren’t changing as we move along the path. That is,

$$w^j (\partial_j v^k) = 0, \quad (26)$$

where  $w^j$  are the components of the vector along the direction in which  $v$  is being dragged. How does this equation look in other co-ordinate systems<sup>9</sup>? Let’s suppose  $\tilde{x}^j$  is some other co-ordinate system. Then we have

$$w^j = (\tilde{\partial}_l x^j) \tilde{w}^l \quad (27)$$

$$\partial_j = (\partial_j \tilde{x}^m) \tilde{\partial}_m \quad (28)$$

$$v^k = (\tilde{\partial}_n x^k) \tilde{v}^n. \quad (29)$$

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<sup>9</sup>We’re going to work out the answer to this question in detail, which involves quite a bit of index juggling and notation. However, not a lot is lost on a first read if you don’t bother following all the manipulations in detail, but just skim the general argument through to the conclusions, which are Eqs. (33) and (34).

Substituting these equations into Eq. (26) we obtain:

$$(\tilde{\partial}_l x^j)(\partial_j \tilde{x}^m) \tilde{w}^l (\tilde{\partial}_m (\tilde{\partial}_n x^k) \tilde{v}^n) = 0. \quad (30)$$

Observe that  $(\tilde{\partial}_l x^j)(\partial_j \tilde{x}^m) = \delta_l^m$ , by elementary calculus. Thus the previous equation can be simplified to

$$\tilde{w}^l (\tilde{\partial}_l (\tilde{\partial}_n x^k) \tilde{v}^n) = 0. \quad (31)$$

This gives us

$$\tilde{w}^l (\tilde{\partial}_n x^k) (\tilde{\partial}_l \tilde{v}^n) + \tilde{w}^l (\tilde{\partial}_{ln}^2 x^k) \tilde{v}^n = 0, \quad (32)$$

where  $\tilde{\partial}_{ln}^2 \equiv \partial^2 / \partial \tilde{x}^l \partial \tilde{x}^n$ . Multiplying the entire equation by  $\partial_k \tilde{x}^m$  and using  $(\partial_k \tilde{x}^m)(\tilde{\partial}_n x^k) = \delta_n^m$ , we obtain

$$\tilde{w}^l (\tilde{\partial}_l \tilde{v}^m) + \tilde{A}_{ln}^m \tilde{w}^l \tilde{v}^n = 0, \quad (33)$$

where the coefficients

$$\tilde{A}_{ln}^m \equiv (\partial_k \tilde{x}^m)(\tilde{\partial}_{ln}^2 x^k) \quad (34)$$

are known as the *connection coefficients* or *Christoffel symbols*. Note that  $\tilde{A}_{ln}^m$  is not changed if the  $l$  and  $n$  indices are interchanged. A convenient shorthand for Eq. (33) is

$$\tilde{w}^l (\tilde{\partial}_l \tilde{v} + \tilde{A}_l \tilde{v}) = 0, \quad (35)$$

where it is understood that  $\tilde{v}$  is a vector, and  $\tilde{A}_l$  is a matrix that can act on that vector, with entries  $\tilde{A}_{ln}^m$ . We'll be using this type of shorthand frequently.

The condition of Eq. (26) was defined with respect to some specific co-ordinate system. Unfortunately, it's quite unwieldy to have to specify a co-ordinate system. An easier way to go is to work in the co-ordinate system of our choice, and then impose the condition of Eq. (33) for some choice of  $A_{ln}^m$  (where we now drop the tildes) which is symmetric in  $l$  and  $n$ . The following exercise shows that this procedure is completely equivalent to the above procedure. From a practical point of view, the condition of Eq. (33) is easier to work with, and this is what we do from now on.

**Exercise 5.12:** Show that for any function  $A_{ln}^m$  symmetric in  $l$  and  $n$  there is a co-ordinate system such that the condition of Eq. (33) is equivalent to the condition of Eq. (26)<sup>10</sup>.

Eq. (33) defines for us what it means for a vector  $v$  not to change as it is dragged in the direction  $w$ . In fact, given any vector  $w$  in  $T_x M$  and a vector field  $v$  defined in

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<sup>10</sup>I haven't done this exercise (I got muddled in the attempt, and have left it for another day) and I'm not one hundred percent sure it's correct.

some neighbourhood of  $x$ , we define the *connection* or *covariant derivative of  $v$  in the direction of  $w$*  by

$$D_w v \equiv w^l (\partial_l v + A_l v), \quad (36)$$

where we again use the convention that  $A_l$  is a matrix that can act on  $v$ , with entries  $A_{ln}^m$ .  $D_w v$  can be thought of as specifying the rate of change of the vector field  $v$  in the  $w$  direction; the result is another vector field,  $D_v w$ .

The approach we've taken to defining the connection is rather unconventional. In the standard account an axiomatic approach to the definition is taken, not unlike the approach we've taken to the exterior derivative in Section 5.5. After some work, the axiomatic approach yields the expression of Eq. (36), and in practice one often thinks of a connection as being determined by specifying the matrices  $A_l$  in some local co-ordinate system at each point on the manifold. In these notes I've pursued an alternative approach based on Eq. (26), since I believe it gives somewhat more motivation for the introduction of the connection.

## 5.7 Vector bundles, general connections, and $G$ -bundles

In physics we are typically interested in many different types of field, including scalar fields, vector fields, tensor fields, spinor fields, and so on. In this section we'll extend the notion of connection defined in the last section in order that we can do calculus with these other types of fields as well.

To do this, we introduce the notion of a *vector bundle*. What we do is attach to each point  $x$  on the manifold a vector space  $V_x$  which we call the *fiber* over  $x$ . We are going to represent the field at the point  $x$  through an element of  $V_x$ . For this reason, we require that the dimensionality of  $V_x$  be the same at every point. We call the resulting collection of objects (manifold plus fibers at each point) a *vector bundle*. It is convenient to have a single notation for a vector bundle, and we'll use  $E$ . Note that for some vector bundles the spaces  $V_x$  are all real vector spaces; for others, they are all complex vector spaces.

To represent fields on this vector bundle, we define a *section*  $s$  to be a function which takes each point  $x$  on the manifold to a vector  $s(x)$  in the fiber  $V_x$ . Although the terminology is a little peculiar from a physicist's point of view, the concept is obviously just that of a field. The temperature field on the surface of a coffee cup is a good example of a section over a vector bundle. In this case, to each point  $x$  on the surface of the coffee cup we have a fiber  $V_x = \mathbb{R}$ . The section therefore assigns a single real number, representing the temperature, to each point on the surface. Another example is a tensor field; examples which may be familiar include the metric and the stress-energy tensor. In this case the fibers  $V_x$  are just the appropriate vector spaces of tensors.

We've already met several examples of vector bundles, although we didn't use the terminology. For example, if to each point  $x$  on the manifold we define the fiber  $V_x$  to be the tangent space  $T_x M$ , then the resulting vector bundle is known as the *tangent bundle*, often denoted  $TM$ . Similarly, if the fiber is  $T_x^* M$ , then the resulting vector bundle is the

*cotangent bundle*, often denoted  $T^*M$ . We can also define a vector bundle whose fibers are the spaces  $\Lambda^p(T_x^*M)$ ; we will denote this vector bundle as  $\Omega^p(M)$ , risking confusion with the space of  $p$ -forms, which are just sections over the vector bundle  $\Omega^p(M)$ . In practice, it should be clear from context which is meant.

**The connection on a general vector bundle:** Just as for vectors in the tangent space, it is by no means obvious how we should identify vectors in one fiber of a vector bundle with vectors in another fiber. Once again, we introduce a notion of covariant differentiation in order to make precise what it means for a section not to be changing as we move in some direction. In particular, if  $w$  is a vector field, and  $s$  is a section, then we will define another section  $D_w s$  over the same vector bundle, representing the rate of change of  $s$  in the  $w$  direction, at any given point on the manifold.

How should we define  $D_w s$ ? By analogy with our earlier analysis for vector fields, let us start again by defining what it means for a section  $s$  not to be changing. The most obvious way of proceeding is to demand that in some co-ordinate representation for the manifold and the fibers, we have

$$w^j \partial_j s^\alpha = 0. \tag{37}$$

Notice that for this equation to make sense, we must choose some co-ordinate representation  $s^\alpha$  for the vector in the fiber, i.e., we must choose a basis for the vector space  $V_x$ . We use the greek index to emphasize that this choice of basis refers to the fiber, while roman indices refer to motion on the manifold. With the fiber co-ordinate representation understood, the above equation can conveniently be rewritten as

$$w^j \partial_j s = 0. \tag{38}$$

Of course, there is some arbitrariness in the way we specify both the co-ordinates on the manifold, and also the co-ordinate representation in the fiber<sup>11</sup>. We are going to change representations to a new representation in terms of co-ordinates  $\tilde{x}^j$  for the manifold and  $\tilde{s}^\alpha$  for the fiber. The resulting line of reasoning is, of course, very similar to that we used earlier for the covariant derivative of a vector field; the impatient reader may wish to skim ahead to the results of our analysis, Eqs. (45) and (46). To make the change of co-ordinates we observe that

$$w^j = (\tilde{\partial}_l x^j) \tilde{w}^l \tag{39}$$

$$\partial_j = (\partial_j \tilde{x}^m) \tilde{\partial}_m \tag{40}$$

$$s = L \tilde{s}. \tag{41}$$

Here,  $L$  is a transformation relating the two co-ordinate systems for the fiber. Since we could have used any basis for the fiber,  $L$  may be an arbitrary invertible linear

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<sup>11</sup>Note that earlier, in our discussion of covariant differentiation of a vector field, once co-ordinates on the manifold were chosen, there was a natural choice for co-ordinates in the tangent space. There is no natural choice for fiber co-ordinates in the case of a general vector bundle. Fortunately, this doesn't affect the reasoning that follows.

transformation. Substituting these relations into Eq. (38) we obtain

$$(\tilde{\partial}_l x^j)(\partial_j \tilde{x}^m) \tilde{w}^l (\tilde{\partial}_m L \tilde{s}) = 0. \quad (42)$$

Observe that  $(\tilde{\partial}_l x^j)(\partial_j \tilde{x}^m) = \delta_l^m$  by elementary calculus. The previous equation can therefore be simplified to

$$\tilde{w}^l (\tilde{\partial}_l L \tilde{s}) = 0 \quad (43)$$

This gives us

$$\tilde{w}^l L (\tilde{\partial}_l \tilde{s}) + \tilde{w}^l (\tilde{\partial}_l L) \tilde{s} = 0. \quad (44)$$

Multiplying by  $L^{-1}$  gives

$$\tilde{w}^l (\tilde{\partial}_l \tilde{s} + \tilde{A}_l \tilde{s}) = 0, \quad (45)$$

where

$$\tilde{A}_l = L^{-1} (\tilde{\partial}_l L). \quad (46)$$

This equation may be rewritten explicitly in terms of the co-ordinate representation for the fiber,

$$\tilde{w}^l (\tilde{\partial}_l \tilde{v}^\alpha + \tilde{A}_{l\beta}^\alpha \tilde{v}^\beta) = 0, \quad (47)$$

where  $\tilde{A}_{l\beta}^\alpha$  are the matrix elements of  $\tilde{A}_l$ . This notation emphasizes the fact that  $\tilde{A}_l$  maps from the fiber  $V_x$  back into the fiber  $V_x$ .

Just as for the case of vector fields, the above analysis suggests that when  $w$  is a vector field and  $s$  is a section over a vector bundle  $E$ , we can define a new section, the covariant derivative  $D_w s$  of  $s$  with respect to  $w$ , by choosing the connection coefficients  $A_{j\beta}^\alpha$  and setting

$$D_w s \equiv w^j \partial_j s + A_j s. \quad (48)$$

This expression defines what we mean by a connection, through the choice of the connection coefficients  $A_j$ ; we shall denote a generic connection by  $D$  or  $D_w$ , and specify it by giving  $A_j$  explicitly in some local co-ordinate system.

**$G$ -bundles and gauge transformations:** Recall from the introduction that Yang-Mills theories are an example of a special type of classical field theory known as a *gauge theory*. The way a gauge theory works is to fix a manifold,  $M$ , and then to consider a special type of vector bundle  $E$  defined over  $M$ , known as a  *$G$ -bundle*. The idea is to choose a fixed group,  $G$ , and a representation,  $V$ , of that group. Then each fiber  $V_x$  is a copy of the representation. We call the resulting structure a  *$G$ -bundle*. A section of a  $G$ -bundle assigns to each point  $x$  on the manifold a group element  $g(x)$  acting on  $V_x$ .

The function  $g(x)$  is sometimes called a *gauge transformation* or *gauge symmetry*, and the group  $G$  the *gauge group*.

As we'll see later in detail, the idea of gauge theory is to find examples of field theories which are invariant under such gauge transformations, in some suitable sense. The Yang-Mills theories we construct are examples of such theories. People sometimes refer to *Abelian* or *non-Abelian* gauge theories. This is a reference to whether the group  $G$  is an Abelian or non-Abelian group. We will see that electromagnetism has an Abelian gauge group, while the standard model has a non-Abelian gauge group. It turns out that the equations of Yang-Mills theory are substantially simpler in the Abelian case, due to the vanishing of some commutators.

## 5.8 Maxwell's equations

The first steps in developing a Yang-Mills theory such as Maxwell's equations are to: (1) pick a manifold, for Maxwell's equations this is just the Minkowski space familiar from special relativity; (2) pick a gauge group, for Maxwell's equations this is just the group  $U(1)$ , i.e., the unit circle in the complex plane, consisting of complex numbers  $z = e^{i\theta}$  satisfying  $|z| = 1$ ; and (3) pick a representation of that group to act as the fiber. For Maxwell's equations we choose  $V_x = \mathbb{C}$ , and so a gauge transformation is just a function  $e^{i\theta(x)}$  which acts by multiplication on a section  $s(x)$ .

With these three steps we have defined a  $G$ -bundle. This  $G$ -bundle is the space on which our Yang-Mills theory will live. A natural guess is that the fundamental physical object in our Yang-Mills theory is going to be a section of that bundle — in the case of Maxwell's equations, that would be a complex function  $s(x)$  on Minkowski space. In fact, quite a different approach is taken. The fundamental physical field in a Yang-Mills theory is the connection,  $D_w$ . This is used to define a second mathematical object, the curvature, and the Yang-Mills equations are a set of equations that constrain the curvature.

How does this procedure connect to the standard description of electromagnetism? In fact, it turns out that the connection coefficients  $A_{j\beta}^\alpha$  play the same role in Yang-Mills theory as the vector potential does in electromagnetism. As a result, the Yang-Mills equations governing the connection  $D_w$  are equivalent to a set of equations for the connection coefficients. With the choice of gauge group and representation described above, these equations are just Maxwell's equations for the vector potential.

To be a little more explicit, in the case of Maxwell's equations the fiber is a one-dimensional vector space. As a result, the coefficients  $\alpha$  and  $\beta$  can be suppressed, and the connection matrix  $A_j$  is just a complex valued function on the manifold, as we would expect for the vector potential of elementary electromagnetism.

## 5.9 The curvature

Suppose  $v$  and  $w$  are vector fields and  $s$  is a section. Given a connection  $D$ , we can define the corresponding *curvature*  $F(v, w)$  by

$$F(v, w)s \equiv D_v D_w s - D_w D_v s - D_{[v, w]}s, \quad (49)$$

where  $[v, w]$  is the commutator of vector fields, c.f. Eq. (7). Thus, the curvature  $F(v, w)$  takes as input a section, and returns as output another section, defined as above. Note, incidentally, that sometimes we'll refer to  $F$  as the curvature, without specifying  $v$  and  $w$ .

Naively, looking at the definition of  $F(v, w)$  it looks as though its action at a given point ought to depend on the value of  $s$  in the neighbourhood of that point. After all, if we compute a covariant derivative like  $D_v s$  then the value at a point  $x$  depends not only on the value of  $s$  at  $x$ , but also how  $s$  varies in the neighbourhood of  $x$ . Remarkably, though, the value of  $F(v, w)s$  at a point  $x$  depends only on the value of  $v, w$  and  $s$  at the point  $x$ , and doesn't depend on their behaviour near  $x$ . We'll prove this below, using a co-ordinate representation for the curvature.

**Exercise 5.13:** Prove that the value of  $D_v s$  at a point  $x$  depends only on the value of  $v$  at that point, and not on the value of  $v$  at nearby points. Prove by explicit example that this is *not* the case for  $s$ , i.e., give an example of two sections  $s$  and  $s'$  which have the same value at  $x$  but for which  $D_v s \neq D_v s'$ .

Why define the curvature? At a purely formal level the definition is straightforward enough — the curvature measures the extent to which taking commutators of covariant derivatives is the same as the covariant derivative in the direction of the commutator. But why define the curvature at all? What properties does it have? In what sense is the above definition related to our intuitive concept of curvature? Unfortunately, understanding curvature in depth is a complex task, and we do not have space here to discuss these questions in the depth they deserve. I refer you to [1] and the references therein for a more in-depth discussion of the meaning of curvature.

For our purposes, we are simply going to regard the definition of curvature as an algebraic fact, and use it to specify the Yang-Mills equations. Let us start by developing an explicit formula for the curvature in co-ordinates. In particular, suppose  $v$  and  $w$  are vector fields on the manifold, and expand them in a local co-ordinate basis,  $v = v^j \partial_j$  and  $w = w^k \partial_k$ . This gives  $F(v, w) = v^j w^k F_{jk}$ , where  $F_{jk} \equiv F(\partial_j, \partial_k)$ . Note that  $F_{jk}$  maps the fiber  $V_x$  back into itself.

We are going to work out a simple formula for the  $F_{jk}$ . To do this, recall that  $[\partial_j, \partial_k] = 0$  and observe that  $D_0 s = 0$ , so we have

$$F_{jk}(s) = (D_j D_k - D_k D_j)(s), \quad (50)$$

where we define  $D_j \equiv D_{\partial_j}$ . Simply applying the definitions for  $D_j$  and  $D_k$  in terms of the connection matrices  $A_j$  then gives

$$F_{jk}s = (\partial_j A_k - \partial_k A_j + [A_j, A_k])s, \quad (51)$$



and since  $s$  was arbitrary we get the explicit formula for  $F_{jk}$ ,

$$F_{jk} = \partial_j A_k - \partial_k A_j + [A_j, A_k]. \quad (52)$$

In the case of electromagnetism the matrices  $A_j$  are  $1 \times 1$  matrices (i.e., complex numbers), and the commutator therefore vanishes, giving

$$F_{jk} = \partial_j A_k - \partial_k A_j. \quad (53)$$

This formula will be familiar to physicists as the *Faraday tensor* for electromagnetism. In a general Yang-Mills theory the curvature  $F$  plays a role analogous to the Faraday tensor in electromagnetism. Interestingly, Yang has recorded [3] that when he first tried generalizing Maxwell's equations to obtain a gauge theory with a non-Abelian gauge group, he worked in terms of a field  $F_{jk}$  defined by Eq. (53), and didn't get anywhere. It wasn't until several years later that he and Mills thought to add the commutator term seen in Eq. (52), and they were able to use this to obtain a non-Abelian gauge theory.

## 5.10 The exterior covariant derivative on bundles

We've defined a notion of the covariant derivative on a vector bundle,  $E$ , and of the exterior derivative acting on space of  $p$ -forms,  $\Omega^p(M)$ . In this section we're going to define the *exterior covariant derivative*, which combines and generalizes these operations.

**$E$ -valued  $p$ -forms:** To achieve this combination, we first need a way of combining vector bundles. Suppose  $E$  and  $F$  are vector bundles with respective fibers  $V_x$  and  $W_x$  at each point  $x$  of the manifold. Then we can form a new bundle  $E \otimes F$  simply by forming the tensor product of fibers, e.g.,  $V_x \otimes W_x$ . In particular,  $E \otimes \Omega^p(M)$  is a vector bundle, where  $\Omega^p(M)$  is the vector bundle with fibers  $\Lambda^p(T_x^*M)$ . We call a section of  $E \otimes \Omega^p(M)$  an  *$E$ -valued  $p$ -form*. The reason for this nomenclature becomes clear if we expand the section locally in co-ordinates as

$$E_{j_1 \dots j_p} \otimes (dx^{j_1} \wedge \dots \wedge dx^{j_p}). \quad (54)$$

We see that just as an ordinary  $p$ -form acts on a tensor product of vector fields to produce a scalar field, an  $E$ -valued  $p$ -form can act in a natural way on a tensor product of vector fields to produce a section of  $E$ .

**The curvature as a vector bundle-valued 2-form:** We have already met an example of a vector bundle-valued 2-form, namely, the curvature. To see that this is the case, we need to define a new vector bundle, known as<sup>12</sup>  $\text{End}(E)$ , whose fibers consist of linear maps of the fibers  $V_x$  into themselves. Thus, a section  $v(x)$  of  $\text{End}(E)$  just assigns to each point  $x$  a linear map  $v(x)$  from  $V_x$  into itself. A good example of a section of  $\text{End}(E)$  is a gauge symmetry.

We can now see that the curvature is an  $\text{End}(E)$ -valued 2-form. In fact, we claim that, with a very mild abuse of notation,

$$F = F_{jk} \otimes dx^j \wedge dx^k. \quad (55)$$

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<sup>12</sup>End stands for endomorphism.

To see that this is the case, let us expand  $v = v^j \partial_j$  and  $w = w^k \partial_k$ , and note that

$$(F_{jk} \otimes dx^j \wedge dx^k)(v \otimes w) = \frac{F_{jk}(v^j w^k - v^k w^j)}{2}. \quad (56)$$

But  $F_{jk} = -F_{kj}$ , and so the right-hand side is equal to  $F_{jk} v^j w^k$ , which is  $F(v, w)$ , completing the proof of the claim.

**Exercise 5.14:** Prove that  $F_{jk} = -F_{kj}$ .

**The exterior covariant derivative:** We are now in position to define the exterior covariant derivative on a vector bundle. Let us expand our  $E$ -valued  $p$ -form as

$$\omega = s_J \otimes dx^J, \quad (57)$$

where  $dx^J$  is short-hand for a suitable wedge product of basis one-forms,  $dx^{j_1} \wedge dx^{j_2} \wedge \dots$ , and the  $s_J$  are corresponding sections. Then we define the exterior covariant derivative by

$$d_D \omega \equiv D_k s_J \otimes dx^k \wedge dx^J, \quad (58)$$

where, as before,  $D_k$  is shorthand for  $D_{\partial_k}$ . Note that this definition is a natural generalization of Eq. (21) to  $E$ -valued  $p$ -forms. A useful shorthand for this formula is

$$d_D = D_k \otimes dx^k \wedge. \quad (59)$$

**Exercise 5.15:** Show that the definition of Eq. (58) is independent of the co-ordinates used to make the definition.

**The connection on  $\text{End}(E)$ :** Recall from Section 5.2 that the first of the Yang-Mills equations is  $d_D F = 0$ . On the face of it we are in position to understand all the elements of this equation — after all, we understand both  $d_D$  and  $F$ . There is actually a minor hurdle we still have to pass. To understand this hurdle, recall that the fundamental field in the Yang-Mills equations is the connection  $D$  over the bundle  $E$ . Thus, the equation  $d_D F = 0$  would make sense if  $F$  was an  $E$ -valued 2-form. In fact,  $F$  is an  $\text{End}(E)$ -valued 2-form, and to make sense of the equation we need a way of defining a covariant derivative  $D$  on the bundle  $\text{End}(E)$ . It turns out that there is a natural way of going from a covariant derivative  $D$  on  $E$  to a corresponding covariant derivative (which we also denote  $D$ ) on  $\text{End}(E)$ .

In particular, suppose we require that the Leibnitz rule hold, so that if  $s$  is a section of  $E$ , and  $T$  is a section of  $\text{End}(E)$ , then we demand

$$D_v(Ts) = (D_v T)(s) + T(D_v s). \quad (60)$$

As noted above, we are using the same notation  $D_v T$  for the covariant derivative on the bundle  $\text{End}(E)$  as on the bundle  $E$ ; these are, of course, quite different objects. In order for this equation to hold, we are forced to define

$$(D_v T)(s) \equiv D_v(Ts) - T(D_v s). \quad (61)$$

This is the desired covariant derivative on  $\text{End}(E)$ . Another way of looking at this is that we have defined

$$D_v T \equiv [D_v, T]. \quad (62)$$

Be careful in interpreting this equation. On the left-hand side  $D_v$  is the covariant derivative on the bundle  $\text{End}(E)$ , while on the right-hand side  $D_v$  is the covariant derivative on the bundle  $E$ .

Eq. (62) is all very well, but despite the notation it is by no means clear that this defines a connection on the bundle  $\text{End}(E)$ . To see that this is the case it helps to compute a more explicit representation for  $D_v T$ . Observe that we have

$$(D_j T)(s) = D_j(Ts) - T D_j s \quad (63)$$

$$= \partial_j(Ts) + A_j Ts - T \partial_j s - T A_j s \quad (64)$$

$$= (\partial_j T + [A_j, T])(s). \quad (65)$$

Thus, we have the beautiful explicit formula in co-ordinates,

$$D_j T = \partial_j T + [A_j, T]. \quad (66)$$

Although we won't go through the details, it ought to be clear from this formula and the linearity of  $[A_j, T]$  in  $T$  that  $D_j$  does define a genuine connection on  $\text{End}(E)$ .

We are now in position to understand the first of the Yang-Mills equations,  $d_D F = 0$ . Summing up, we have a connection,  $D$ , defined on sections of a  $G$ -bundle  $E$ . We use this connection to: (1) define the curvature,  $F$ , which is an  $\text{End}(E)$ -valued 2-form; and (2) to define a connection on  $\text{End}(E)$ . This latter connection is then used to define an exterior covariant derivative  $d_D$ , which is applied to  $F$ , giving rise to an  $\text{End}(E)$ -valued 3-form, which we require to vanish.

### 5.11 The Bianchi identity

Recall from electromagnetism that once we've defined the vector potential, two of Maxwell's equations are identities that don't carry any physical content beyond the definition of the Faraday tensor in terms of the vector potential. Analogously, the first Yang-Mills equation,  $d_D F = 0$ , is really an identity known as the *Bianchi identity*, and follows directly from the definition of the curvature in terms of the connection  $D$ . To see that this is the case, observe that in local co-ordinates

$$d_D F = D_j F_{kl} \otimes dx^j \wedge dx^k \wedge dx^l \quad (67)$$

$$= \frac{D_j F_{kl} + D_k F_{lj} + D_l F_{jk}}{3} \otimes dx^j \wedge dx^k \wedge dx^l, \quad (68)$$

where in the second line we have made use of the antisymmetry of  $dx^j \wedge dx^k \wedge dx^l$ . Recalling Eq. (62) and the definition  $F_{kl} = [D_k, D_l]$ , we see that this equation is equivalent to

$$d_D F = \frac{[D_j, [D_k, D_l]] + [D_k, [D_l, D_j]] + [D_l, [D_j, D_k]]}{3} \otimes dx^j \wedge dx^k \wedge dx^l. \quad (69)$$

But  $D_j, D_k$  and  $D_l$  are all just linear operators, and it is simple algebra to prove that the Jacobi identity  $[D_j, [D_k, D_l]] + [D_k, [D_l, D_j]] + [D_l, [D_j, D_k]] = 0$  holds for all such triples of linear operators. It follows that the Bianchi identity  $d_D F = 0$  holds identically.

**Exercise 5.16:** Prove the Jacobi identity  $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$  for an arbitrary triple of linear operators,  $X, Y$  and  $Z$ .

## 5.12 The Hodge-\* operation

The next ingredient in Yang-Mills theory is the Hodge-\* operation, which is a map that converts  $p$ -forms into  $(n - p)$ -forms, where  $n$  is the dimension of the manifold. To understand the definition of the Hodge-\* it helps to momentarily turn away from differential geometry, and suppose that we are working in a vector space,  $V$ , that has an inner product. Suppose  $e^1, \dots, e^n$  are an orthonormal basis for this space. Then we define

$$*(e^{j_1} \wedge \dots \wedge e^{j_p}) \equiv \pm e^{j_{p+1}} \wedge \dots \wedge e^{j_n}, \quad (70)$$

where  $j_{p+1}, \dots, j_n$  are the integers in the list  $1, \dots, n$  (in order) which are not in the list  $j_1, \dots, j_p$ , and where the sign is determined by the parity of the permutation taking  $1, \dots, n$  to  $j_1, \dots, j_n$ . The Hodge-\* is defined for a general vector in  $\Lambda^p(V)$  by extending this definition linearly. We'll see below that it is well-defined and unique, i.e., it does not depend on the choice of basis.

In applications to physics and Yang-Mills theory, we wish to extend the definition of the Hodge-\* so that it can be applied to  $p$ -forms  $\omega \in \Omega^p(M)$ . To do this, we need some notion of an inner product on the cotangent space  $T_x^*(M)$ , so that we can define what it means for a set of vectors in  $T_x^*M$  to be orthonormal. Such a notion is provided by equipping the manifold with a Riemannian metric, and we assume this has been done. If you're not familiar with Riemannian geometry, do not be concerned — what is important here is that we have some sensible notion of orthonormal cotangent vectors for each cotangent space  $T_x^*M$ . Once this has been done, the Hodge-\* operation can be defined pointwise for an arbitrary  $p$ -form  $\omega$ .

In fact, in applications to physics there is one further wrinkle, connected with the notion of inner product. You may recall that in special relativity the metric can be thought of as an inner product function with the strange property that lengths can sometimes be *negative*. For this reason, in Minkowski space the notion of orthonormality

needs to be extended slightly. A similar extension must be done on  $T_x^*M$  in general, and we define a basis set  $e^j$  for  $T_x^*M$  to be orthonormal if their inner product satisfies

$$\langle e^j, e^k \rangle = \epsilon(j)\delta^{jk}, \quad (71)$$

where  $\epsilon(j) = \pm 1$ . With this extension, we define

$$*(e^{j_1} \wedge \dots \wedge e^{j_p}) \equiv \pm e^{j_{p+1}} \wedge \dots \wedge e^{j_n}, \quad (72)$$

where  $j_{p+1}, \dots, j_n$  are again the integers in the list  $1, \dots, n$  (in order) which are not in the list  $j_1, \dots, j_p$ , and where the sign is determined by the parity of the permutation taking  $1, \dots, n$  to  $j_1, \dots, j_n$ , and *also* by the product  $\epsilon(j_1) \dots \epsilon(j_p)$ . The extension to arbitrary  $p$ -forms is again made by linearity.

**Well-definedness of the Hodge-\***: With the definition made above, it is clear that the Hodge- $*$  is a type of complement operation. What is less obvious is that it is well-defined, and unique. One way of seeing that is to observe that it satisfies

$$\alpha \wedge (*\beta) = \langle \alpha, \beta \rangle \text{vol}, \quad (73)$$

where the volume form  $\text{vol} \equiv e^1 \wedge \dots \wedge e^n$ . To verify this equation is a straightforward computation. All that remains then is to verify that the volume form is well-defined, i.e., does not depend on the choice of basis  $e^1, \dots, e^n$ . Note that the definition of the Hodge- $*$  depends upon the metric; change the metric and we change the volume form.

**Exercise 5.17:** Show that the volume form  $\text{vol}$  is well-defined.

**Examples:** Let's work in ordinary three-dimensional space, with the usual inner product. We can identify  $dx, dy$  and  $dz$  as an orthonormal basis, and we have:

$$*(dx) = dy \wedge dz \quad (74)$$

$$*(dy) = -dx \wedge dz \quad (75)$$

$$*(dz) = dx \wedge dy. \quad (76)$$

It is straightforward to verify that  $*^2$  is the identity in this case. If  $v$  and  $w$  are one-forms then calculations show that:

$$*(v \wedge w) = v \times w \quad (77)$$

$$*(dv) = \text{curl } v \quad (78)$$

$$*d*v = \text{div } v. \quad (79)$$

Thus, one way of viewing the Hodge- $*$  is as a natural way of generalizing the definitions of the cross product, curl and div operations.

**Extending the Hodge- $*$  to  $E$ -valued  $p$ -forms:** We've defined the Hodge- $*$  operation for ordinary  $p$ -forms. In Yang-Mills theory we'll need to extend the definition of

the Hodge- $*$  to  $E$ -valued  $p$ -forms. We do this in the most straightforward way possible, simply allowing the  $*$  to act on that part of the tensor product containing  $\Lambda^p(V)$ . In terms of co-ordinates, if  $\omega = \omega_J \otimes dx^J$  is an  $E$ -valued  $p$ -form then we define

$$*\omega \equiv \omega_J \otimes *(dx^J). \quad (80)$$

**Problem for the author 5.1:** Why is the Hodge- $*$  operation defined? Is there a natural mathematical question that can be used to motivate the introduction of the Hodge- $*$  operation?

### 5.13 The current

In the standard elementary description of Maxwell's equations we describe charge using both a charge density  $\rho$  and a current vector  $\vec{j}$ . In the more sophisticated relativistic approach, these objects are combined into a *current four-vector*. When we reformulate Maxwell's equations as a Yang-Mills theory, it is most convenient to use the metric to convert the current four-vector into a dual vector, which we denote  $J$ .

The current  $J$  appearing in the general Yang-Mills equations generalizes this description, with  $J$  now being an  $\text{End}(E)$ -valued one-form. We can therefore expand  $J$  as

$$J = J_k \otimes dx^k. \quad (81)$$

If we are working in Minkowski space, then the value  $J_0$  can be thought of as a generalized charge, and  $J_k$  (for  $k \neq 0$ ) as a generalized charge flux in the  $x^k$  direction.

### 5.14 The Yang-Mills equations

We are now in position to understand all the elements in the Yang-Mills equations:

$$d_D F = 0 \quad (82)$$

$$*d_D *F = J. \quad (83)$$

We have seen that the first of these equations, Eq. (82), is really an identity known as the *Bianchi identity*, which follows automatically from the definition of  $F$  in terms of the covariant derivative. This generalizes the way two of Maxwell's equations follow from the definition of the electric and magnetic fields in terms of the vector potential. The definition of the curvature  $F$  in terms of the covariant derivative may be stated as

$$F(v, w)s \equiv ([D_v, D_w] - D_{[v, w]})(s). \quad (84)$$

Equivalently, in terms of the (generalized) vector potential  $A_j$ , the curvature may be given the expression

$$F = F_{jk} \otimes dx^j \wedge dx^k; \quad F_{jk} = \partial_j A_k - \partial_k A_j + [A_j, A_k]. \quad (85)$$

The second of the Yang-Mills equations, Eq. (83), may be given the explicit co-ordinate representation

$$D_i F_{jk} \otimes *(dx^i \wedge *(dx^j \wedge dx^k)) = J. \quad (86)$$

Recall also the expression for the covariant derivative in terms of the vector potential,

$$D_i F_{jk} = \partial_i F_{jk} + [A_i, F_{jk}]. \quad (87)$$

Eqs. (85) and (86) are an explicit co-ordinate representation for the Yang-Mills equation in terms of the vector potential  $A_j$ , with the covariant derivative defined as in Eq. (87). These equations make it obvious that the Yang-Mills equations are a set of second order ordinary differential equations for the vector potential. In the general case these equations are nonlinear. However, in the special case of abelian gauge theory the commutators vanish, and the equations become linear equations for the vector potential.

**Exercise 5.18:** Show how to deduce Maxwell's equations from Eqs. (85)-(87).

**Problem for the author 5.2:** One thing I'm not yet sure on, and need to check: for the commutators to vanish, it must be that the vector potential  $A_j$  is constrained to lie in the Lie algebra of the gauge group. If this is true, then so too will  $F_{jk}$ , and all the above statements are correct. Similar constraints apply to the current.

**Initial value problem:** We saw earlier that the Maxwell equations need to be supplemented by the Lorentz force law if initial conditions are to determine the later configuration. In a similar way, the initial conditions for a solution to the Yang-Mills equations do not necessarily determine the later configuration. To do that requires a supplementary equation analogous to the Lorentz force law, telling us how moving charges accelerate in response to applied fields. We won't describe such an equation here.

However, there is a notable way in which the Yang-Mills equations alone do specify a well-defined initial value problem. Suppose we simply regard as fixed the value of the current  $J$  across all of spacetime, and the curvature  $F_{jk}$  at some initial time. Then the Yang-Mills equations completely determine the value of  $\partial_0 F_{jk}$  for all  $j \neq k$ , and can, in principle, be integrated to determine the configuration of  $F_{jk}$  at later times. Thus, there is a sense in which knowing the initial values for the fields uniquely determines later configurations, provided one assumes that the current is given.

**Gauge invariance:** One of the motivations for the Yang-Mills equations is that they satisfy gauge symmetry. Recall that a gauge symmetry is described by a section  $g(x)$  of the  $G$ -bundle on which we are working. Suppose we have a solution  $(D, J)$  to the Yang-Mills equations. Then I claim that  $(D', J')$  is another solution to the Yang-Mills equations, where  $D'$  and  $J'$  are related to  $D$  and  $J$  by the gauge symmetry as follows:

$$D' = g D g^{-1} \quad (88)$$

$$J' = g J g^{-1}. \quad (89)$$

Note that  $J$  is an  $\text{End}(E)$ -valued 2-form, and  $g$  of course acts only on the  $\text{End}(E)$  part, i.e., if  $J = J_{kl}dx^k \wedge dx^l$  then  $gJg^{-1} = gJ_{kl}g^{-1}dx^k \wedge dx^l$ .

We'll prove that  $D'$  and  $J'$  satisfy the Yang-Mills equations below. Before doing that, let us talk about the mathematical and physical motivation for wanting gauge symmetry to be satisfied.

Mathematically, the transformation of Eqs. (88) and (89) are both natural consequences of the transformation  $s(x) \rightarrow s'(x) = g(x)s(x)$  of sections. In particular, with this map on sections, we see that:

$$D's' = (Ds)'; \quad J's' = (Js)'. \quad (90)$$

Less obvious is the fact that  $D'$  is itself a covariant derivative. This can be seen by working in co-ordinates:

$$D'_v(s) = gv^j(\partial_j(g^{-1}s) + A_jg^{-1}s) \quad (91)$$

$$= v^j(\partial_j s + [g(\partial_j g^{-1}) + gA_jg^{-1}]s). \quad (92)$$

We see from this expression that  $D'$  is another connection with connection coefficients given by  $g(\partial_j g^{-1}) + gA_jg^{-1}$ .

Physically, there is of course no ironclad *a priori* reason why Eqs. (88) and (89) should be symmetries of the theory. Nonetheless, the hypothesis that they are has proven to be remarkably fruitful. Much of twentieth century physics can be summed up by the idea of first searching for symmetries, and then trying to construct the simplest possible theory which satisfies that symmetry. The *stronger* the symmetry we impose, the more constrained is the resulting class of theories to be considered. Gauge symmetry is a very constraining symmetry, and so the hypothesis of gauge invariance greatly assists in cutting down the number of theories to be considered.

Let's now prove that Eqs. (88) and (89) are indeed symmetries of the Yang-Mills equations, Eqs. (82) and (83). The case of Eq. (82) follows immediately from the Bianchi identity, since  $D'$  is a connection and  $F'$  is the corresponding curvature. To prove that Eq. (83) is satisfied, observe first the transformation law for the curvature,  $F_{kl}$ . We have:

$$F'_{kl} = [D'_k, D'_l] \quad (93)$$

$$= g[D_k, D_l]g^{-1} \quad (94)$$

$$= gF_{kl}g^{-1}. \quad (95)$$



Thus we have

$$*d_{D'} * F' = D'_j F'_{kl} \otimes *(dx^j \wedge *(dx^k \wedge dx^l)) \quad (96)$$

$$= [D'_j, F'_{kl}] \otimes *(dx^j \wedge *(dx^k \wedge dx^l)) \quad (97)$$

$$= [gD_j g^{-1}, gF_{kl} g^{-1}] \otimes *(dx^j \wedge *(dx^k \wedge dx^l)) \quad (98)$$

$$= g[D_j, F_{kl}] g^{-1} \otimes *(dx^j \wedge *(dx^k \wedge dx^l)) \quad (99)$$

$$= g(*d_D * F) g^{-1} \quad (100)$$

$$= gJg^{-1} \quad (101)$$

$$= J', \quad (102)$$

which is the desired result. Note that in moving from the first to the second line of this derivation we changed from using the connection  $D$  defined on the vector bundle  $\text{End}(E)$  to the connection  $D$  on the vector bundle  $E$ .

## References

- [1] John Baez and Javier P. Muniain. *Gauge Fields, Knots And Gravity*. World Scientific, Singapore, 1994.
- [2] Gerardus 't Hooft, editor. *50 Years of Yang-Mills theory*. World Scientific, New Jersey, 2005.
- [3] C. N. Yang. Gauge Invariance and Interactions. In [2], 2005.