Proof
Let d be the dimension of system A. Note that

$$S\left(\rho^{AB} \left\| \frac{I}{d} \otimes \rho^B\right.\right) = -S(A,B) - \operatorname{tr}\left(\rho_{AB} \log\left(\frac{I}{d} \otimes \rho^B\right)\right) \tag{11.104}$$

$$= -S(A,B) - \operatorname{tr}(\rho^B \log \rho^B) + \log d \tag{11.105}$$

$$= -S(A|B) + \log d. \tag{11.106}$$

Thus $S(A|B) = \log d - S(\rho^{AB}\|I/d \otimes \rho^B)$. The concavity of $S(A|B)$ follows from the joint convexity of the relative entropy. □

Theorem 11.14: (**Strong subadditivity**) For any trio of quantum systems, A, B, C, the inequalities

$$S(A) + S(B) \leq S(A,C) + S(B,C) \tag{11.107}$$

$$S(A,B,C) + S(B) \leq S(A,B) + S(B,C) \tag{11.108}$$

hold.

Proof
The two inequalities are in fact equivalent. We will use concavity of the conditional entropy to prove the first, and then show that the second follows. Define a function $T(\rho^{ABC})$ of density operators on the system ABC,

$$T(\rho^{ABC}) \equiv S(A) + S(B) - S(A,C) - S(B,C) = -S(C|A) - S(C|B). \tag{11.109}$$

From the concavity of the conditional entropy we see that $T(\rho^{ABC})$ is a convex function of ρ^{ABC}. Let $\rho^{ABC} = \sum_i p_i |i\rangle\langle i|$ be a spectral decomposition of ρ^{ABC}. From the convexity of T, $T(\rho^{ABC}) \leq \sum_i p_i T(|i\rangle\langle i|)$. But $T(|i\rangle\langle i|) = 0$ as for a pure state $S(A,C) = S(B)$ and $S(B,C) = S(A)$. It follows that $T(\rho^{ABC}) \leq 0$, and thus

$$S(A) + S(B) - S(A,C) - S(B,C) \leq 0, \tag{11.110}$$

which is the first inequality we set out to prove.

To obtain the second inequality, introduce an auxiliary system R purifying the system ABC. Then using the just-proved inequality we have

$$S(R) + S(B) \leq S(R,C) + S(B,C). \tag{11.111}$$

Since $ABCR$ is a pure state, $S(R) = S(A,B,C)$ and $S(R,C) = S(A,B)$, so (11.111 becomes

$$S(A,B,C) + S(B) \leq S(A,B) + S(B,C), \tag{11.112}$$

as we set out to show. □

Exercise 11.24: We obtained strong subadditivity as a consequence of the inequality $S(A) + S(B) \leq S(A,C) + S(B,C)$. Show that this inequality can be obtained as a consequence of strong subadditivity.

Exercise 11.25: We obtained strong subadditivity as a consequence of the concavity of the conditional entropy, $S(A|B)$. Show that the concavity of the conditional