for some constant d, establishing Equation (A3.6). (*Comment*: for practical purposes it may be interesting to obtain good bounds on d.)

Exercise A3.3: Let $\vec{x}$ and $\vec{y}$ be any two real vectors. Show that

$$D(u(\vec{x}), u(\vec{y})) = 2\sqrt{2}\sqrt{1 - \cos(x/2)\cos(y/2) - \sin(x/2)\sin(y/2)\hat{x}\cdot\hat{y}}\,, \tag{A3.10}$$

where $x \equiv \|\vec{x}\|, y \equiv \|\vec{y}\|$, and $\hat{x}$ and $\hat{y}$ are unit vectors in the $\vec{x}$ and $\vec{y}$ directions, respectively.

Exercise A3.4: Show that in the case $\vec{y} = 0$ the formula for $D(u(\vec{x}), u(\vec{y}))$ reduces to

$$D(u(\vec{x}), I) = 4\sin\left|\frac{x}{4}\right|. \tag{A3.11}$$

Exercise A3.5: Show that when $x, y \leq \epsilon$,

$$D(u(\vec{x}), u(\vec{y})) = \|\vec{x} - \vec{y}\| + O(\epsilon^3). \tag{A3.12}$$

Proof
(Lemma A3.2)

Suppose $\mathcal{G}_l$ is an ϵ^2-net in S_ϵ. The first step of the proof is to show that $[\mathcal{G}_l, \mathcal{G}_l]_{\text{gp}}$ is a $C\epsilon^3$-net for S_{ϵ^2} and some constant C.

Let $U \in S_{\epsilon^2}$ and pick $\vec{x}$ such that $U = u(\vec{x})$. By Exercise A3.4 it follows that $x \leq \epsilon^2 + O(\epsilon^6)$. Choose any pair of vectors $\vec{y}$ and $\vec{z}$ of length at most $\epsilon + O(\epsilon^5)$ such that $\vec{x} = \vec{y} \times \vec{z}$. $\mathcal{G}_l$ is an ϵ^2-net for S_ϵ, so choose U_1 and U_2 in $\mathcal{G}_l \cap S_\epsilon$ such that

$$D(U_1, u(\vec{y})) < \epsilon^2 + O(\epsilon^5) \tag{A3.13}$$
$$D(U_2, u(\vec{z})) < \epsilon^2 + O(\epsilon^5), \tag{A3.14}$$

and let $\vec{y}_0$ and $\vec{z}_0$ be chosen such that $U_1 = u(\vec{y}_0)$ and $U_2 = u(\vec{z}_0)$. By Exercise A3.4 it follows that $y_0, z_0 \leq \epsilon + O(\epsilon^3)$. Our goal is to show that $D(U, [U_1, U_2]_{\text{gp}})$ is smaller than $C\epsilon^3$. To do this, we use the triangle inequality,

$$D(U, [U_1, U_2]_{\text{gp}}) \leq D(U, u(\vec{y}_0 \times \vec{z}_0)) + D(u(\vec{y}_0 \times \vec{z}_0), [U_1, U_2]_{\text{gp}}). \tag{A3.15}$$

The second term is at most $d'\epsilon^3$, by Exercise A3.2, where d' is a constant slightly larger than d because of the possible contribution due to the fact that $y_0, z_0 < \epsilon + O(\epsilon^3)$, rather than $y_0, z_0 < \epsilon$. Substituting $U = u(\vec{x})$, making use of Exercise A3.5, introducing an appropriate constant d'' and doing some elementary algebra gives

$$\begin{aligned}
D(U, [U_1, U_2]_{\text{gp}}) &\leq D(u(\vec{x}), u(\vec{y}_0 \times \vec{z}_0)) + d'\epsilon^3 && \text{(A3.16)}\\
&= \|\vec{x} - \vec{y}_0 \times \vec{z}_0\| + d''\epsilon^3 && \text{(A3.17)}\\
&= \|\vec{y} \times \vec{z} - \vec{y}_0 \times \vec{z}_0\| + d''\epsilon^3. && \text{(A3.18)}\\
&= \|[(\vec{y} - \vec{y}_0) + \vec{y}_0] \times [(\vec{z} - \vec{z}_0) + \vec{z}_0] - \vec{y}_0 \times \vec{z}_0\| + d''\epsilon^3 && \text{(A3.19)}\\
&\leq (d'' + 2)\epsilon^3 + O(\epsilon^4) && \text{(A3.20)}\\
&\leq C\epsilon^3\,, && \text{(A3.21)}
\end{aligned}$$

where C is an appropriately chosen constant.

The second step in the proof of the lemma is to apply a translation step like that used in the main part of the proof of the Solovay–Kitaev theorem. Specifically, given